

# SOLUTIONS TO A SINGULARLY PERTURBED SUPERCRITICAL ELLIPTIC EQUATION ON A RIEMANNIAN MANIFOLD CONCENTRATING AT A SUBMANIFOLD

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ABSTRACT. Given a smooth Riemannian manifold  $(\mathcal{M}, g)$  we investigate the existence of positive solutions to the equation

$$-\varepsilon^2 \Delta_g u + u = u^{p-1} \quad \text{on } \mathcal{M}$$

which concentrate at some submanifold of  $\mathcal{M}$  as  $\varepsilon \rightarrow 0$ , for supercritical nonlinearities. We obtain a positive answer for some manifolds, which include warped products.

Using one of the projections of the warped product or some harmonic morphism, we reduce this problem to a problem of the form

$$-\varepsilon^2 \operatorname{div}_h (c(x) \nabla_h u) + a(x)u = b(x)u^{p-1},$$

with the same exponent  $p$ , on a Riemannian manifold  $(M, h)$  of smaller dimension, so that  $p$  turns out to be subcritical for this last problem. Then, applying Lyapunov-Schmidt reduction, we establish existence of a solution to the last problem which concentrates at a point as  $\varepsilon \rightarrow 0$ .

## 1. INTRODUCTION

Let  $(\mathfrak{M}, \mathfrak{g})$  be a compact smooth Riemannian manifold, without boundary, of dimension  $m \geq 2$ . We consider the problem

$$(\wp_\varepsilon) \quad \begin{cases} -\varepsilon^2 \Delta_{\mathfrak{g}} v + v = v^{p-1}, \\ v \in H_{\mathfrak{g}}^1(\mathfrak{M}), \quad v > 0. \end{cases}$$

where  $p > 2$  and  $\varepsilon^2$  is a singular perturbation parameter. The space  $H_{\mathfrak{g}}^1(\mathfrak{M})$  is the completion of  $C^\infty(\mathfrak{M})$  with respect to the norm defined by  $\|v\|_{\mathfrak{g}}^2 := \int_{\mathfrak{M}} (|\nabla_{\mathfrak{g}} v|^2 + v^2) d\mu_{\mathfrak{g}}$ .

Let  $2_m^* := \infty$  if  $m = 2$  and  $2_m^* := \frac{2m}{m-2}$  if  $m \geq 3$  be the critical Sobolev exponent in dimension  $m$ . In the subcritical case, where  $p < 2_m^*$ , solutions to  $(\wp_\varepsilon)$  which concentrate at a point are known to exist. In [9] Byeon and Park showed that there are solutions with one peak concentrating at a maximum point of the scalar curvature of  $(\mathfrak{M}, \mathfrak{g})$  as  $\varepsilon \rightarrow 0$ . Single-peak solutions concentrating at a stable critical point of the scalar curvature of  $(\mathfrak{M}, \mathfrak{g})$  as  $\varepsilon \rightarrow 0$  were obtained by Micheletti and Pistoia in [22], whereas in [13] Dancer, Micheletti and Pistoia proved the existence

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solutions with  $k$  peaks which concentrate at an isolated minimum of the scalar curvature of  $(\mathfrak{M}, g)$  as  $\varepsilon \rightarrow 0$ . Some results on sign changing solutions, as well as multiplicity results, are also available, see [23, 7, 10, 29, 18, 17].

We are specially interested in the critical and the supercritical case, where  $p \geq 2_m^*$ , and in solutions exhibiting concentration at positive dimensional submanifolds of  $\mathfrak{M}$  as  $\varepsilon \rightarrow 0$ .

For the analogous equation

$$(1.1) \quad -\varepsilon^2 \Delta v + V(x)v = v^{p-1} \quad \text{in } \Omega,$$

in a bounded smooth domain  $\Omega$  in  $\mathbb{R}^n$  with Dirichlet or Neumann boundary conditions, or in the entire space  $\mathbb{R}^n$ , there is a vast literature concerning solutions concentrating at one or at a finite set of points. Various results concerning concentration at an  $(n-1)$ -dimensional sphere are nowadays also available, see e.g. [2, 3, 4, 5, 8, 21] and the references therein.

A fruitful approach to produce solutions to equation (1.1) which concentrate at other positive dimensional manifolds, for  $p$  up to some supercritical exponent, is to reduce it to an equation in a domain of lower dimension. This approach was introduced by Ruf and Srikanth in [26], where the reduction is given by a Hopf map. Reductions may also be performed by means of other maps which preserve the laplacian, or by considering rotational symmetries, or by a combination of both, as has been recently done in [1, 11, 12, 19, 20, 24, 25, 30] for different problems.

Next, we describe some of these reductions in the context of Riemannian manifolds. For simplicity, from now on we consider all manifolds to be smooth, compact and without boundary.

**1.1. Harmonic morphisms.** Let  $(\mathfrak{M}, g)$  and  $(M, g)$  be Riemannian manifolds of dimensions  $m$  and  $n$  respectively. A *harmonic morphism* is a horizontally conformal submersion  $\pi : \mathfrak{M} \rightarrow M$  with dilation  $\lambda : \mathfrak{M} \rightarrow [0, \infty)$  which satisfies the equation

$$(1.2) \quad (n-2)\mathcal{H}(\nabla_g \ln \lambda) + (m-n)\kappa^\mathcal{V} = 0,$$

where  $\kappa^\mathcal{V}$  is the mean curvature of the fibers and  $\mathcal{H}$  is the projection of the tangent space of  $\mathfrak{M}$  onto the space orthogonal to the fibers of  $\pi$ , see [6, 15, 16, 28]. Typical examples are the Hopf fibrations

$$\mathbb{S}^m \rightarrow \mathbb{R}P^m, \quad \mathbb{S}^{2m+1} \rightarrow \mathbb{C}P^m, \quad \mathbb{S}^{4m+3} \rightarrow \mathbb{H}P^m, \quad \mathbb{S}^{15} \rightarrow \mathbb{S}^8,$$

which are Riemannian submersions (i.e.  $\lambda \equiv 1$ ) with totally geodesic fibers  $\mathbb{S}^0$ ,  $\mathbb{S}^1$ ,  $\mathbb{S}^3$  and  $\mathbb{S}^7$  respectively, see [6, Examples 2.4.14-2.4.17]. So they trivially satisfy (1.2).

The main property of a harmonic morphism is that it preserves the Laplace-Beltrami operator, i.e. it satisfies

$$\Delta_g(u \circ \pi) = \lambda^2 [(\Delta_g u) \circ \pi]$$

for every  $\mathcal{C}^2$ -function  $u : M \rightarrow \mathbb{R}$ . The following proposition is an immediate consequence of this fact.

**Proposition 1.1.** *Let  $\pi : \mathfrak{M} \rightarrow M$  be a harmonic morphism with dilation  $\lambda$ . Assume there exists  $\mu : M \rightarrow (0, \infty)$  such that  $\mu \circ \pi = \lambda^2$ . Then  $u : M \rightarrow \mathbb{R}$  solves*

$$(1.3) \quad -\varepsilon^2 \Delta_g u + \frac{1}{\mu(x)} u = \frac{1}{\mu(x)} u^{p-1} \quad \text{on } M$$

iff  $v := u \circ \pi : \mathfrak{M} \rightarrow \mathbb{R}$  solves

$$(1.4) \quad -\varepsilon^2 \Delta_{\mathfrak{g}} v + v = v^{p-1} \quad \text{on } \mathfrak{M}.$$

Note that the exponent  $p$  is the same in both equations. If  $n < m$  then  $2_m^* < 2_n^*$ , so  $p \in [2_m^*, 2_n^*)$  is subcritical for problem (1.3) but it is critical or supercritical for problem (1.4). Moreover, if solutions  $u_\varepsilon$  of (1.3) concentrate at some point  $x_0 \in M$  the corresponding solutions  $v_\varepsilon := u_\varepsilon \circ \pi$  to problem (1.4) concentrate at  $\pi^{-1}(x_0)$ , which is a manifold of dimension  $m - n$ .

**1.2. Warped products.** A different type of reduction can be performed on warped products. If  $(M, g)$  and  $(N, h)$  are Riemannian manifolds of dimensions  $n$  and  $k$  respectively, and  $f : M \rightarrow (0, \infty)$  is a smooth function, the *warped product*  $M \times_{f^2} N$  is the cartesian product  $M \times N$  equipped with the Riemannian metric  $g + f^2 h$ .

The projection  $\pi_N : M \times_{f^2} N \rightarrow N$  is a harmonic morphism with dilation  $\lambda(y, z) = \frac{1}{f(y)}$  [6, Example 2.4.26] but there is no function  $\mu : M \rightarrow (0, \infty)$  such that  $\mu \circ \pi = \lambda^2$  unless  $f$  is constant. So we cannot apply Proposition 1.1. Instead, we consider the projection  $\pi_M : M \times_{f^2} N \rightarrow M$ . A straightforward computation gives the following result, cf. [14].

**Proposition 1.2.**  $u : M \rightarrow \mathbb{R}$  solves

$$(1.5) \quad -\varepsilon^2 \operatorname{div}_g (f^k(x) \nabla_g u) + f^k(x) u = f^k(x) u^{p-1} \quad \text{on } M$$

iff  $v := u \circ \pi_M : M \times_{f^2} N \rightarrow \mathbb{R}$  solves

$$(1.6) \quad -\varepsilon^2 \Delta_{\mathfrak{g}} v + v = v^{p-1} \quad \text{on } M \times_{f^2} N.$$

The exponent  $p$  is again the same in both equations. So, if  $k > 0$ , then  $p \in [2_{n+k}^*, 2_n^*)$  is subcritical for problem (1.5) but it is critical or supercritical for problem (1.6). Moreover, if solutions  $u_\varepsilon$  of (1.5) concentrate at some point  $x_0 \in M$ , then the solutions  $v_\varepsilon := u_\varepsilon \circ \pi_M$  to problem (1.6) concentrate at  $\pi_M^{-1}(x_0) \cong (N, f^2(x_0)h)$ .

**1.3. The reduced problem.** Propositions 1.1 and 1.2 lead to study the problem

$$(1.7) \quad \begin{cases} -\varepsilon^2 \operatorname{div}_g (c(x) \nabla_g u) + a(x) u = b(x) u^{p-1} \\ u \in H_g^1(M), \quad u > 0. \end{cases}$$

where  $(M, g)$  is an  $n$ -dimensional compact smooth Riemannian manifold without boundary,  $n \geq 2$ ,  $p \in (2, 2_n^*)$ ,  $a, b, c$  are positive real-valued  $\mathcal{C}^2$ -functions on  $M$ , and  $\varepsilon$  is a positive parameter.

In order to state our main result we need the following definition.

**Definition 1.** Let  $\phi \in \mathcal{C}^1(M, \mathbb{R})$ . A set  $K \subset \{x \in M : \nabla_g \phi(x) = 0\}$  is a  $\mathcal{C}^1$ -stable critical set for  $\phi$  if for each  $\mu > 0$  there exists  $\delta > 0$  such that, if  $\psi \in \mathcal{C}^1(M, \mathbb{R})$  with

$$\max_{d_g(x, K) \leq \mu} |\phi(x) - \psi(x)| + |\nabla_g \phi(x) - \nabla_g \psi(x)| \leq \delta,$$

then  $\psi$  has a critical point  $x_0$  with  $d_g(x_0, K) \leq \mu$ . Here  $d_g$  denotes the geodesic distance associated to the Riemannian metric  $g$ .

We shall prove the following result.

**Theorem 1.3.** Let  $K$  be a  $\mathcal{C}^1$ -stable critical set for the function

$$\Gamma(x) := \frac{c(x)^{\frac{n}{2}} a(x)^{\frac{p}{p-2} - \frac{n}{2}}}{b(x)^{\frac{2}{p-2}}}.$$

Then, for every  $\varepsilon$  small enough, problem (1.7) has a solution  $u_\varepsilon$  which concentrates at a point  $x_0 \in K$  as  $\varepsilon \rightarrow 0$ .

This result, together with Propositions 1.1 and 1.2, implies the existence of solutions to problem  $(\wp_\varepsilon)$  which concentrate at a submanifold for subcritical, critical and supercritical exponents. More precisely, the following results hold true.

**Corollary 1.4.** *Assume there exists a harmonic morphism  $\pi : \mathfrak{M} \rightarrow M$  whose dilation  $\lambda : \mathfrak{M} \rightarrow (0, \infty)$  is of the form  $\lambda^2 = \mu \circ \pi$ . Let  $n := \dim M \geq 2$  and let  $K$  be a  $C^1$ -stable critical set for the function  $\mu^{\frac{n-2}{2}} : M \rightarrow (0, \infty)$ . Then, for every  $p \in (2, 2_n^*)$  and  $\varepsilon$  small enough, problem (1.7) has a solution  $v_\varepsilon$  which concentrates at the fiber  $\pi^{-1}(x_0)$  over some point  $x_0 \in K$ , as  $\varepsilon \rightarrow 0$ .*

**Corollary 1.5.** *Let  $\mathfrak{M}$  be the warped product  $M \times_{f^2} N$ ,  $n := \dim M \geq 2$ ,  $k := \dim N$  and  $K$  be a  $C^1$ -stable critical set for the function  $f^k : M \rightarrow (0, \infty)$ . Then for every  $p \in (2, 2_n^*)$  and  $\varepsilon$  small enough, problem (1.7) has a solution  $v_\varepsilon$  which concentrates at  $\{x_0\} \times N$ , for some point  $x_0 \in K$ , as  $\varepsilon \rightarrow 0$ .*

**1.4. Surfaces of revolution.** The following application illustrates how one can combine these results to produce solutions which concentrate at different submanifolds.

**Theorem 1.6.** *Let  $M$  be a compact smooth Riemannian submanifold of  $\mathbb{R}^\ell \times (0, \infty)$ , without boundary,  $n := \dim M \geq 1$ , and let*

$$\mathfrak{M} := \{(y, z) \in \mathbb{R}^\ell \times \mathbb{R}^{k+1} : (y, |z|) \in M\},$$

*with the Riemannian metric inherited from  $\mathbb{R}^{\ell+k+1}$ . The following statements hold true:*

- (a) *For each  $p \in (2, 2_{n+k}^*)$  problem  $(\wp_\varepsilon)$  has a solution  $v_\varepsilon^0$  which concentrates at two points in  $\mathfrak{M}$  as  $\varepsilon \rightarrow 0$ .*
- (b) *If  $k$  is odd then, for each  $p \in (2, 2_{n+k}^*)$  and each  $j \in \mathbb{N}$ , problem  $(\wp_\varepsilon)$  has a solution  $v_\varepsilon^{0,j}$  which concentrates at  $j$  points in  $\mathfrak{M}$  as  $\varepsilon \rightarrow 0$ .*
- (c) *If  $k \geq 3$  is odd then, for each  $p \in (2, 2_{n+k-1}^*)$ , problem  $(\wp_\varepsilon)$  has a solution  $v_\varepsilon^1$  which concentrates at a 1-dimensional sphere in  $\mathfrak{M}$  as  $\varepsilon \rightarrow 0$ .*
- (d) *If  $k = 4m + 3$  then, for each  $p \in (2, 2_{n+k-3}^*)$ , problem  $(\wp_\varepsilon)$  has a solution  $v_\varepsilon^3$  which concentrates at a 3-dimensional sphere in  $\mathfrak{M}$  as  $\varepsilon \rightarrow 0$ .*
- (e) *If  $k = 15$  then, for each  $p \in (2, 2_{n+8}^*)$ , problem  $(\wp_\varepsilon)$  has a solution  $v_\varepsilon^7$  which concentrates at a 7-dimensional sphere in  $\mathfrak{M}$  as  $\varepsilon \rightarrow 0$ .*
- (f) *If  $n \geq 2$  then, for each  $p \in (2, 2_n^*)$ , problem  $(\wp_\varepsilon)$  has a solution  $v_\varepsilon^k$  which concentrates at a  $k$ -dimensional sphere in  $\mathfrak{M}$  as  $\varepsilon \rightarrow 0$ .*

*Proof.* Observe that  $\mathfrak{M}$  is isometric to the warped product  $M \times_{f^2} \mathbb{S}^k = (M \times \mathbb{S}^k, g + f^2 h)$  where  $g$  is the euclidean metric on  $M$ ,  $h$  is the standard metric on  $\mathbb{S}^k$  and  $f : M \rightarrow (0, \infty)$  is the projection  $f(y, t) := t$ . The isometry is given by

$$\mathfrak{M} \ni (y, z) \mapsto \left( (y, |z|), \frac{z}{|z|} \right) \in M \times_{f^2} \mathbb{S}^k.$$

(a): The product  $\pi : M \times_{f^2} \mathbb{S}^k \rightarrow M \times_{f^2} \mathbb{R}P^k$  of the identity map on  $M$  with the Hopf fibration is a Riemannian submersion whose fiber is diffeomorphic to  $\mathbb{S}^0$ . Hence, it is a harmonic morphism with dilation  $\lambda \equiv 1$ . Theorem 1.2 in [22] yields a positive solution  $u_\varepsilon^0 \in H_g^1(M \times_{f^2} \mathbb{R}P^k)$  to equation (1.3) with  $\mu \equiv 1$ , which

concentrates at a  $\mathcal{C}^1$ -stable critical point  $\xi_0$  of the scalar curvature of  $M \times_{f^2} \mathbb{R}P^k$ , for every  $p \in (2, 2_{n+k}^*)$ . By Proposition 1.1,  $v_\varepsilon := u_\varepsilon \circ \pi$  is a solution to  $(\phi_\varepsilon)$  which concentrates at the pair of points  $\pi^{-1}(\xi_0)$  as  $\varepsilon \rightarrow 0$ .

(b): If  $k$  is odd we identify  $\mathbb{R}^{k+1}$  with  $\mathbb{C}^{\frac{k+1}{2}}$ . The group  $\Gamma_j := \{e^{2\pi i s/j} : s = 0, \dots, j-1\}$  acts by multiplication on each coordinate of  $\mathbb{C}^{\frac{k+1}{2}}$ . Since this action is free on  $\mathbb{S}^k$ , the orbit space  $\mathbb{S}^k/\Gamma_j$  is a smooth manifold. We endow  $\mathbb{S}^k/\Gamma_j$  with the Riemannian metric which turns the orbit map  $\mathbb{S}^k \rightarrow \mathbb{S}^k/\Gamma_j$  into a Riemannian submersion. Now we can argue as in (a), replacing the Hopf map by this orbit map.

(c): Let  $k = 2m + 1$  and  $\mathfrak{N} := (M \times \mathbb{C}P^m, f^{\frac{2}{d-2}}g + f^{\frac{2}{d-2}+2}\tilde{h})$  where  $d := 2m + n = \dim(M \times \mathbb{C}P^m)$  and  $\tilde{h}$  is the standard metric on  $\mathbb{C}P^m$ . If we identify  $\mathfrak{M}$  with  $(M \times \mathbb{S}^{2m+1}, g + f^2h)$ , the product of the identity map on  $M$  with the Hopf fibration  $\mathbb{S}^{2m+1} \rightarrow \mathbb{C}P^m$  is a horizontally conformal submersion  $\pi : \mathfrak{M} \rightarrow \mathfrak{N}$  with dilation  $\lambda(y, z) = f(y, |z|)^{\frac{1}{d-2}} = |z|^{\frac{1}{d-2}}$ , whose fiber is diffeomorphic to  $\mathbb{S}^1$ . The mean curvature of the fibers is  $\kappa^\mathcal{V}(y, z) = -\frac{z}{|z|^2}$ . Therefore,  $\lambda$  satisfies equation (1.2) and, consequently,  $\pi$  is a harmonic morphism. Moreover,  $\lambda^2 = \mu \circ \pi$ , where  $\mu((y, t), \zeta) = t^{\frac{2}{d-2}}$  for  $(y, t) \in M$ ,  $\zeta \in \mathbb{C}P^m$ . Corollary 1.4 implies that, for every  $p \in (2, 2_d^*)$  and  $\varepsilon$  small enough, problem (1.7) has a solution  $v_\varepsilon^1$  which concentrates at some fiber of  $\pi$  over a  $\mathcal{C}^1$ -stable critical set for the function  $((y, t), \zeta) \mapsto t$  as  $\varepsilon \rightarrow 0$ .

(d): Let  $\mathfrak{N} := (M \times \mathbb{H}P^m, f^{\frac{6}{d-2}}g + f^{\frac{6}{d-2}+2}\tilde{h})$  where  $d := 4m + n = \dim(M \times \mathbb{H}P^m)$  and  $\tilde{h}$  is the standard metric on  $\mathbb{H}P^m$ . If we identify  $\mathfrak{M}$  with  $(M \times \mathbb{S}^{4m+3}, g + f^2h)$ , the product of the identity map on  $M$  with the Hopf fibration  $\mathbb{S}^{4m+3} \rightarrow \mathbb{H}P^m$  is a horizontally conformal submersion  $\pi : \mathfrak{M} \rightarrow \mathfrak{N}$  with dilation  $\lambda(y, z) = |z|^{\frac{3}{d-2}}$ , whose fiber is diffeomorphic to  $\mathbb{S}^3$ . The mean curvature of the fibers is  $\kappa^\mathcal{V}(y, z) = -\frac{z}{|z|^2}$ . Therefore,  $\lambda$  satisfies equation (1.2) and  $\pi$  is a harmonic morphism. Corollary 1.4 implies that, for every  $p \in (2, 2_d^*)$  and  $\varepsilon$  small enough, problem (1.7) has a solution  $v_\varepsilon^3$  which concentrates at some fiber of  $\pi$  over a  $\mathcal{C}^1$ -stable critical set for the function  $((y, t), \zeta) \mapsto t^3$  as  $\varepsilon \rightarrow 0$ .

(e): The proof is similar to that of (c) and (d), this time using the Hopf fibration  $\mathbb{S}^{15} \rightarrow \mathbb{S}^8$ .

(f): This follows immediately from Corollary 1.5.  $\square$

Note that the proof contains information on the location of the sets of concentration. We have recently learned that a similar statement for annuli in  $\mathbb{R}^N$  was proved by Ruf and Srikanth [27].

The rest of this paper is devoted to the proof of Theorem 1.3, which is based on the well-known Lyapunov-Schmidt reduction. The outline of the paper is as follows: In Section 2 we discuss the limit problem. In Section 3 we outline the Lyapunov-Schmidt procedure and use it to prove Theorem 1.3. In Section 4 we establish the finite dimensional reduction, and in Section 5 we obtain the expansion of the reduced functional. We collect some technical facts in Appendix A.

## 2. THE LIMIT PROBLEM

Let  $(M, g)$  be an  $n$ -dimensional compact smooth Riemannian manifold without boundary,  $n \geq 2$ ,  $a, b, c$  be positive real-valued  $\mathcal{C}^2$ -functions on  $M$ , and  $\varepsilon$  be a positive parameter.

We denote by  $H_\varepsilon$  the Sobolev space  $H_g^1(M)$  with the scalar product

$$\langle u, v \rangle_\varepsilon := \frac{1}{\varepsilon^{n-2}} \int_M c(x) \nabla_g u \nabla_g v d\mu_g + \frac{1}{\varepsilon^n} \int_M a(x) u v d\mu_g$$

and norm

$$\|u\|_\varepsilon := \langle u, u \rangle_\varepsilon^{1/2} = \frac{1}{\varepsilon^{n-2}} \int_M c(x) |\nabla_g u|^2 d\mu_g + \frac{1}{\varepsilon^n} \int_M a(x) u^2 d\mu_g.$$

Similarly, we denote by  $L_\varepsilon^q$  be the Lebesgue space  $L_g^q(M)$  endowed with the norm

$$|u|_{q,\varepsilon} := \left( \frac{1}{\varepsilon^n} \int_M |u|^q d\mu_g \right)^{1/q}.$$

We recall that  $|u|_{q,\varepsilon} \leq C \|u\|_\varepsilon$  for  $q \in [2, 2_n^*]$ , where the constant  $C$  does not depend on  $\varepsilon$ .

Fix  $p \in (2, 2_n^*)$  and set

$$A(x) := \frac{a(x)}{c(x)}, \quad B(x) := \frac{b(x)}{c(x)}, \quad \gamma(\xi) := \left( \frac{a(\xi)}{b(\xi)} \right)^{\frac{1}{p-2}}.$$

For  $\xi_0 \in M$  let  $V = V^{\xi_0}$  be the unique positive spherically symmetric solution to

$$(2.1) \quad -c(\xi_0) \Delta V + a(\xi_0) V = b(\xi_0) V^{p-1} \quad \text{in } \mathbb{R}^n.$$

This is the limit equation for problem (1.7) in the tangent space  $T_{\xi_0} M \equiv \mathbb{R}^n$ . It is equivalent to

$$(2.2) \quad -\Delta V + A(\xi_0) V = B(\xi_0) V^{p-1} \quad \text{in } \mathbb{R}^n.$$

A simple computation shows that

$$(2.3) \quad V^{\xi_0}(z) = \gamma(\xi_0) U(\sqrt{A(\xi_0)} z),$$

where  $U$  is the unique positive spherically symmetric solution of

$$(2.4) \quad -\Delta U + U = U^{p-1} \quad \text{in } \mathbb{R}^n.$$

Fix  $r > 0$  smaller than the injectivity radius of  $M$ . Then, the exponential map  $\exp_\xi : B(0, r) \rightarrow B_g(\xi, r)$  is a diffeomorphism for every  $\xi \in M$ . Here the tangent space  $T_\xi M$  is identified with  $\mathbb{R}^n$ ,  $B(0, r)$  is the ball of radius  $r$  in  $\mathbb{R}^n$  centered at 0, and  $B_g(\xi, r)$  denotes the ball of radius  $r$  in  $M$  centered at  $\xi$  with respect to the distance induced by the Riemannian metric  $g$ . Let  $\chi \in C^\infty(\mathbb{R}^n)$  be a radial cut-off function such that  $\chi(z) = 1$  if  $|z| \leq r/2$  and  $\chi(z) = 0$  if  $|z| \geq r$ . For  $\xi \in M$  and  $\varepsilon > 0$  we define  $W_{\varepsilon,\xi} \in H_g^1(M)$  by

$$W_{\varepsilon,\xi}(x) := \begin{cases} V^\xi \left( \frac{1}{\varepsilon} \exp_\xi^{-1}(x) \right) \chi \left( \exp_\xi^{-1}(x) \right) & \text{if } x \in B_g(\xi, r), \\ 0 & \text{otherwise.} \end{cases}$$

Setting  $V_\varepsilon(z) := V \left( \frac{z}{\varepsilon} \right)$  and  $y := \exp_\xi^{-1} x$  we have that

$$W_{\varepsilon,\xi}(\exp_\xi(y)) = V^\xi \left( \frac{y}{\varepsilon} \right) \chi(y) = V_\varepsilon^\xi(y) \chi(y),$$

so the function  $W_{\varepsilon,\xi}$  is simply the function  $V^\xi$  rescaled, cut off and read in a normal neighborhood of  $\xi$  in  $M$ .

Similarly, for  $i = 1, \dots, n$  we define

$$Z_{\varepsilon,\xi}^i(x) := \begin{cases} \psi_\xi^i \left( \frac{1}{\varepsilon} \exp_\xi^{-1}(x) \right) \chi \left( \exp_\xi^{-1}(x) \right) & \text{if } x \in B_g(\xi, r), \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\psi_\xi^i(\eta) = \frac{\partial}{\partial \eta_i} V^\xi(\eta) = \gamma(\xi) \sqrt{A(\xi)} \frac{\partial U}{\partial \eta_i}(\sqrt{A(\xi)} \eta).$$

The functions  $\psi_\xi^i$  are solutions to the linearized equation

$$(2.5) \quad -\Delta \psi + A(\xi) \psi = (p-1)B(\xi) (V^\xi)^{p-2} \psi \quad \text{in } \mathbb{R}^n.$$

Next, we compute the derivatives of  $W_{\varepsilon, \xi}$  with respect to  $\xi$  in a normal neighborhood. Fix  $\xi_0 \in M$ . We write the points  $\xi \in B_g(\xi_0, r)$  as

$$\xi = \xi(y) := \exp_{\xi_0}(y) \quad \text{with } y \in B(0, r),$$

and consider the function

$$\mathcal{E}(y, x) := \exp_{\xi(y)}^{-1}(x) = \exp_{\exp_{\xi_0}(y)}^{-1}(x)$$

defined on the set  $\{(y, x) : y \in B(0, r), x \in B_g(\xi(y), r)\}$ . Then we can write

$$\begin{aligned} W_{\varepsilon, \xi(y)}(x) &= \gamma(\xi(y)) U_\varepsilon(\sqrt{A(\xi(y))} \exp_{\xi(y)}^{-1}(x)) \chi(\exp_{\xi(y)}^{-1}(x)) \\ &= \tilde{\gamma}(y) U_\varepsilon(\sqrt{\tilde{A}(y)} \mathcal{E}(y, x)) \chi(\mathcal{E}(y, x)) \end{aligned}$$

where  $\tilde{A}(y) := A(\exp_{\xi_0}(y))$  and  $\tilde{\gamma}(y) := \gamma(\exp_{\xi_0}(y))$ . Thus, we have

$$\begin{aligned} (2.6) \quad \frac{\partial}{\partial y_1} W_{\varepsilon, \xi(y)} &= \left( \frac{\partial}{\partial y_1} \tilde{\gamma}(y) \right) U_\varepsilon(\sqrt{\tilde{A}(y)} \mathcal{E}(y, x)) \chi(\mathcal{E}(y, x)) \\ &\quad + \tilde{\gamma}(y) U_\varepsilon(\sqrt{\tilde{A}(y)} \mathcal{E}(y, x)) \frac{\partial \chi}{\partial z_k}(\mathcal{E}(y, x)) \frac{\partial}{\partial y_1} \mathcal{E}_k(y, x) \\ &\quad + \frac{1}{\varepsilon} \tilde{\gamma}(y) \chi(\mathcal{E}(y, x)) \frac{\partial}{\partial z_k} \left( U_\varepsilon(\sqrt{\tilde{A}(y)} \mathcal{E}(y, x)) \right) \frac{\partial}{\partial y_1} \mathcal{E}_k(y, x). \end{aligned}$$

One has the Taylor expansions

$$(2.7) \quad \frac{\partial}{\partial y_h} \mathcal{E}_k(0, \exp_{\xi_0} \varepsilon z) = -\delta_{hk} + O(\varepsilon^2 |z|^2),$$

$$(2.8) \quad g^{ij}(\varepsilon z) = \delta_{ij} + \frac{\varepsilon^2}{2} \sum_{r,k=1}^n \frac{\partial^2 g^{ij}}{\partial z_r \partial z_k}(0) z_r z_k + O(\varepsilon^3 |z|^3) = \delta_{ij} + o(\varepsilon),$$

$$(2.9) \quad |g(\varepsilon z)|^{\frac{1}{2}} = 1 - \frac{\varepsilon^2}{4} \sum_{i,r,k=1}^n \frac{\partial^2 g^{ii}}{\partial z_r \partial z_k}(0) z_r z_k + O(\varepsilon^3 |z|^3) = 1 + o(\varepsilon),$$

where, as usual,

$$(g^{ij}(z)) \text{ is the inverse matrix of } (g_{ij}(z)) \quad \text{and} \quad |g(z)| := \det(g_{ij}(z)).$$

**Proposition 2.1.** *There exists a positive constant  $C$  such that*

$$(Z_{\varepsilon, \xi}^h, Z_{\varepsilon, \xi}^k)_\varepsilon = C \delta_{hk} + o(1).$$

*Proof.* Using the Taylor expansions of  $g^{ij}(\varepsilon z)$ ,  $|g(\varepsilon z)|^{\frac{1}{2}}$ ,  $a(\exp_\xi(\varepsilon z))$  and  $c(\exp_\xi(\varepsilon z))$  we obtain

$$\begin{aligned}
\langle Z_{\varepsilon,\xi}^h, Z_{\varepsilon,\xi}^k \rangle_\varepsilon &= \frac{1}{\varepsilon^n} \int_M \varepsilon^2 c(x) \nabla_g Z_{\varepsilon,\xi}^h(x) \nabla_g Z_{\varepsilon,\xi}^k(x) + a(x) Z_{\varepsilon,\xi}^h(x) Z_{\varepsilon,\xi}^k(x) d\mu_g \\
&= \int_{B(0,r/\varepsilon)} \sum_{ij} c(\exp_\xi(\varepsilon z)) g_\xi^{ij}(\varepsilon z) \frac{\partial}{\partial z_i} (\psi_\xi^h(z) \chi(\varepsilon z)) \frac{\partial}{\partial z_j} (\psi_\xi^h(z) \chi(\varepsilon z)) |g_\xi(\varepsilon z)|^{\frac{1}{2}} dz \\
&\quad + \int_{B(0,r/\varepsilon)} a(\exp_\xi(\varepsilon z)) \psi_\xi^h(z) \psi_\xi^k(z) \chi^2(\varepsilon z) dz \\
&= c(\xi) \int_{\mathbb{R}^n} \nabla \psi_\xi^h \nabla \psi_\xi^k dz + a(\xi) \int_{\mathbb{R}^n} \psi_\xi^h \psi_\xi^k dz + o(1) \\
&= C\delta_{hk} + o(1),
\end{aligned}$$

as claimed.  $\square$

### 3. OUTLINE OF THE PROOF OF THEOREM 1.3

Let

$$K_{\varepsilon,\xi} := \text{span} \{ Z_{\varepsilon,\xi}^1, \dots, Z_{\varepsilon,\xi}^n \}$$

and

$$K_{\varepsilon,\xi}^\perp := \left\{ \phi \in H_\varepsilon : \langle \phi, Z_{\varepsilon,\xi}^i \rangle_\varepsilon = 0, i = 1, \dots, n \right\}$$

be its orthogonal complement in  $H_\varepsilon$ . We denote the orthogonal projections onto these subspaces by

$$\Pi_{\varepsilon,\xi} : H_\varepsilon \rightarrow K_{\varepsilon,\xi} \quad \text{and} \quad \Pi_{\varepsilon,\xi}^\perp : H_\varepsilon \rightarrow K_{\varepsilon,\xi}^\perp.$$

Let  $i_\varepsilon^* : L_\varepsilon^{p'} \rightarrow H_\varepsilon$  be the adjoint operator of the Sobolev embedding  $i_\varepsilon : H_\varepsilon \rightarrow L_\varepsilon^p$ . It is well known that

$$(3.1) \quad \|i_\varepsilon^*(v)\|_\varepsilon \leq C_1 |v|_{p',\varepsilon} \quad \forall v \in L_g^{p'},$$

$$(3.2) \quad |u|_{p,\varepsilon} \leq C_2 \|u\|_\varepsilon \quad \forall u \in H_\varepsilon,$$

where the constants  $C_1, C_2$  do not depend on  $\varepsilon$ .

We look for a solution to problem (1.7) of the form

$$u_\varepsilon = W_{\varepsilon,\xi} + \phi \quad \text{with } \phi \in K_{\varepsilon,\xi}^\perp.$$

This is equivalent to solving the pair of equations

$$(3.3) \quad \Pi_{\varepsilon,\xi}^\perp \{ W_{\varepsilon,\xi} + \phi - i_\varepsilon^*(b(x)f(W_{\varepsilon,\xi} + \phi)) \} = 0,$$

$$(3.4) \quad \Pi_{\varepsilon,\xi} \{ W_{\varepsilon,\xi} + \phi - i_\varepsilon^*(b(x)f(W_{\varepsilon,\xi} + \phi)) \} = 0,$$

where, to simplify notation, we have set  $f(u) := (u^+)^{p-1}$ .

Next we state the results needed to prove our main result.

**Proposition 3.1.** *There exist  $\varepsilon_0 > 0$  and  $C > 0$  such that, for each  $\xi \in M$  and each  $\varepsilon \in (0, \varepsilon_0)$ , there exists a unique  $\phi_{\varepsilon,\xi} \in K_{\varepsilon,\xi}^\perp$  which solves (3.3). It satisfies*

$$\|\phi_{\varepsilon,\xi}\|_\varepsilon < C\varepsilon.$$

Moreover,  $\xi \mapsto \phi_{\varepsilon,\xi}$  is a  $\mathcal{C}^1$ -map.

*Proof.* The proof will be given in Section 4.  $\square$



A solution to problem (1.7) is a critical point of the energy functional  $J_\varepsilon : H_\varepsilon \rightarrow \mathbb{R}$  given by

$$J_\varepsilon(u) = \frac{1}{2}\|u\|_\varepsilon^2 - \frac{1}{p\varepsilon^n} \int_M b(x)(u^+)^p d\mu_g.$$

Proposition 3.1 allows us to define the reduced energy functional  $\tilde{J}_\varepsilon : M \rightarrow \mathbb{R}$  as

$$\tilde{J}_\varepsilon(\xi) := J_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}).$$

It has the following property.

**Proposition 3.2.** *If  $\xi_0 \in M$  is a critical point of  $\tilde{J}_\varepsilon$ , then the function  $W_{\varepsilon,\xi_0} + \phi_{\varepsilon,\xi_0}$  is a solution of problem (1.7).*

*Proof.* Set  $\xi = \xi(y) = \exp_{\xi_0}(y)$ . If  $\xi_0$  is a critical point for  $\tilde{J}_\varepsilon$  we have

$$\left. \frac{\partial}{\partial y_h} \tilde{J}_\varepsilon(\exp_{\xi_0}(y)) \right|_{y=0} = 0 \quad \text{for all } h = 1, \dots, n.$$

Since  $\phi_{\varepsilon,\xi(y)}$  solves (3.3), we get

$$\begin{aligned} \frac{\partial}{\partial y_h} \tilde{J}_\varepsilon(\exp_{\xi_0}(y)) &= J'_\varepsilon(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) \left[ \frac{\partial}{\partial y_h} (W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) \right] \\ &= \left\langle W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)} - i_\varepsilon^* (b(x)f(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)})), \frac{\partial}{\partial y_h} (W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) \right\rangle_\varepsilon \\ &= \sum_{l=1}^n C_\varepsilon^l \left\langle Z_{\varepsilon,\xi(y)}, \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} + \frac{\partial}{\partial y_h} \phi_{\varepsilon,\xi(y)} \right\rangle_\varepsilon \end{aligned}$$

for some  $C_\varepsilon^1, \dots, C_\varepsilon^n \in \mathbb{R}$ . We want to prove that, for  $y = 0$ ,  $C_\varepsilon^1 = \dots = C_\varepsilon^n = 0$ . Since  $\phi_{\varepsilon,\xi(y)} \in K_{\varepsilon,\xi(y)}^\perp$  we have

$$\begin{aligned} \left\langle Z_{\varepsilon,\xi(y)}, \frac{\partial}{\partial y_h} \phi_{\varepsilon,\xi(y)} \right|_{y=0} \right\rangle_\varepsilon &= - \left\langle \frac{\partial}{\partial y_h} Z_{\varepsilon,\xi(y)} \right|_{y=0}, \phi_{\varepsilon,\xi(y)} \right\rangle_\varepsilon \\ &= O \left( \left\| \frac{\partial}{\partial y_h} Z_{\varepsilon,\xi(y)} \right|_{y=0} \right\|_\varepsilon \cdot \|\phi_{\varepsilon,\xi(y)}\|_\varepsilon \right) = O(1) \end{aligned}$$

by Lemma A.1 and Proposition 3.1. Moreover, by Lemma A.2 we have

$$\left\langle Z_{\varepsilon,\xi_0}^l, \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right|_{y=0} \right\rangle_\varepsilon = -\frac{1}{\varepsilon} C \delta_{hl} + o\left(\frac{1}{\varepsilon}\right),$$

thus

$$0 = \sum_{l=1}^n C_\varepsilon^l \left\langle Z_{\varepsilon,\xi(y)}, \frac{\partial}{\partial y_h} \phi_{\varepsilon,\xi(y)} \right|_{y=0} + \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right|_{y=0} \right\rangle_\varepsilon = -\frac{C}{\varepsilon} \sum_{l=1}^n C_\varepsilon^l (\delta_{hl} + o(1))$$

and this implies that  $C_\varepsilon^l = 0$  for all  $l = 1, \dots, n$ .  $\square$

**Proposition 3.3.** *The reduced energy is given by*

$$\tilde{J}_\varepsilon(\xi) = \left( \frac{p-2}{2p} \int_{\mathbb{R}^n} U^p dz \right) \frac{c(\xi)^{\frac{n}{2}} a(\xi)^{\frac{p}{p-2} - \frac{n}{2}}}{b(\xi)^{\frac{2}{p-2}}} + O(\varepsilon),$$

$\mathcal{C}^1$ -uniformly with respect to  $\xi$  as  $\varepsilon \rightarrow 0$ .

*Proof.* The proof follows from two lemmas which we prove in Section 5: Lemma 5.2 asserts that  $\tilde{J}_\varepsilon(\xi) = J_\varepsilon(W_{\varepsilon,\xi}) + O(\varepsilon)$  as  $\varepsilon \rightarrow 0$ ,  $\mathcal{C}^1$ -uniformly with respect to  $\xi$ . Then, in Lemma 5.1, we obtain the expansion of  $J_\varepsilon(W_{\varepsilon,\xi})$  which yields the claim.  $\square$

Using the previous propositions we prove our main result.

*Proof of Theorem 1.3.* If  $K$  is a  $\mathcal{C}^1$ -stable critical set for  $\Gamma(\xi) = \frac{c(\xi)^{\frac{n}{2}} a(\xi)^{\frac{p}{p-2} - \frac{n}{2}}}{b(\xi)^{\frac{2}{p-2}}}$  then, by Definition 1 and Proposition 3.3,  $\tilde{J}_\varepsilon$  has a critical point  $\xi_\varepsilon \in M$  such that  $d_g(\xi_\varepsilon, K) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Proposition 3.2 asserts that  $u_\varepsilon := W_{\varepsilon,\xi_\varepsilon} + \Phi_{\varepsilon,\xi_\varepsilon}$  is a solution of (1.7).  $\square$

#### 4. THE FINITE DIMENSIONAL REDUCTION

In this section we solve equation (3.3). We introduce the linear operator

$$\begin{aligned} L_{\varepsilon,\xi} &: K_{\varepsilon,\xi}^\perp \rightarrow K_{\varepsilon,\xi}^\perp \\ L_{\varepsilon,\xi}(\phi) &:= \Pi_{\varepsilon,\xi}^\perp \{ \phi - i_\varepsilon^* [b(\cdot) f'(W_{\varepsilon,\xi}) \phi] \}. \end{aligned}$$

Equation (3.3) can be rewritten as

$$L_{\varepsilon,\xi}(\phi) = N_{\varepsilon,\xi}(\phi) + R_{\varepsilon,\xi},$$

where  $N_{\varepsilon,\xi}$  is the nonlinear term

$$N_{\varepsilon,\xi}(\phi) := \Pi_{\varepsilon,\xi}^\perp \{ i_\varepsilon^* [b(\cdot) (f(W_{\varepsilon,\xi} + \phi) - f(W_{\varepsilon,\xi}) - f'(W_{\varepsilon,\xi}) \phi)] \}$$

and  $R_{\varepsilon,\xi}$  is the remainder

$$R_{\varepsilon,\xi} := \Pi_{\varepsilon,\xi}^\perp \{ i_\varepsilon^* [b(\cdot) f(W_{\varepsilon,\xi})] - W_{\varepsilon,\xi} \}.$$

**Lemma 4.1.** *There exist  $\varepsilon_0$  and  $C > 0$  such that, for every  $\xi \in M$  and  $\varepsilon \in (0, \varepsilon_0)$ ,*

$$\|L_{\varepsilon,\xi}(\phi)\|_\varepsilon \geq C \|\phi\|_\varepsilon \quad \text{for all } \phi \in K_{\varepsilon,\xi}^\perp.$$

*Proof.* Arguing by contradiction, assume there exist sequences  $\varepsilon_k \rightarrow 0$ ,  $\xi_k \in M$  with  $\xi_k \rightarrow \xi \in M$ , and  $\phi_k \in K_{\varepsilon_k,\xi_k}^\perp$  with  $\|\phi_k\|_{\varepsilon_k} = 1$ , such that

$$L_{\varepsilon_k,\xi_k}(\phi_k) =: \psi_k \quad \text{satisfies} \quad \|\psi_k\|_{\varepsilon_k} \rightarrow 0 \text{ as } k \rightarrow +\infty.$$

Let  $\zeta_k \in K_{\varepsilon_k,\xi_k}$  be such that

$$(4.1) \quad \phi_k - i_{\varepsilon_k}^* [b(\cdot) f'(W_{\varepsilon_k,\xi_k}) \phi_k] = \psi_k + \zeta_k.$$

Next, we prove that  $\|\zeta_k\|_{\varepsilon_k} \rightarrow 0$  as  $k \rightarrow +\infty$ . Let  $\zeta_k = \sum_{j=1}^n \alpha_j^k Z_{\varepsilon_k,\xi_k}^j$ . Multiplying (4.1) by  $Z_{\varepsilon_k,\xi_k}^h$  and noting that  $\phi_k, \psi_k \in K_{\varepsilon_k,\xi_k}^\perp$  we obtain

$$\begin{aligned} \sum_{j=1}^n \alpha_j^k \left\langle Z_{\varepsilon_k,\xi_k}^j, Z_{\varepsilon_k,\xi_k}^h \right\rangle_{\varepsilon_k} &= - \left\langle i_{\varepsilon_k}^* [b(\cdot) f'(W_{\varepsilon_k,\xi_k}) \phi_k], Z_{\varepsilon_k,\xi_k}^h \right\rangle_{\varepsilon_k} \\ (4.2) \quad &= - \frac{1}{\varepsilon_k^n} \int_M b(x) f'(W_{\varepsilon_k,\xi_k}) \phi_k Z_{\varepsilon_k,\xi_k}^h d\mu_g. \end{aligned}$$

Set

$$\tilde{\phi}_k := \begin{cases} \phi_k (\exp_{\xi_k}(\varepsilon_k z)) \chi(\varepsilon_k z) & \text{if } z \in B(0, r/\varepsilon_k), \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to prove that  $\|\tilde{\phi}_k\|_{H^1(\mathbb{R}^n)} \leq C\|\phi_k\|_{\varepsilon_k} \leq C$  for some positive constant  $C$ . Thus, there exists  $\tilde{\phi} \in H^1(\mathbb{R}^n)$  such that, up to a subsequence,  $\tilde{\phi}_k \rightarrow \tilde{\phi}$  weakly in  $H^1(\mathbb{R}^n)$  and strongly in  $L^p_{\text{loc}}(\mathbb{R}^n)$  for all  $p \in (2, 2_n^*)$ . Since  $\phi_k \in K_{\varepsilon_k, \xi_k}^\perp$  we get

$$\begin{aligned}
\sum_{j=1}^n \alpha_j^k \left\langle Z_{\varepsilon_k, \xi_k}^j, Z_{\varepsilon_k, \xi_k}^h \right\rangle_{\varepsilon_k} &= -\frac{1}{\varepsilon_k^n} \int_M b(x) f'(W_{\varepsilon_k, \xi_k}) \phi_k Z_{\varepsilon_k, \xi_k}^h d\mu_g \\
&= \left\langle \phi_k, Z_{\varepsilon_k, \xi_k}^h \right\rangle_{\varepsilon_k} - \frac{1}{\varepsilon_k^n} \int_M b(x) f'(W_{\varepsilon_k, \xi_k}) \phi_k Z_{\varepsilon_k, \xi_k}^h d\mu_g \\
&= \frac{1}{\varepsilon_k^n} \int_M \left[ \varepsilon_k^2 c(x) \nabla \phi_k \nabla Z_{\varepsilon_k, \xi_k}^h + a(x) Z_{\varepsilon_k, \xi_k}^h \phi_k - b(x) f'(W_{\varepsilon_k, \xi_k}) \phi_k Z_{\varepsilon_k, \xi_k}^h \right] d\mu_g \\
&= \int_{B(0, r/\varepsilon_k)} \sum_{l, m=1}^n g^{lm} c(\exp_{\xi_k}(\varepsilon_k z)) (\varepsilon_k z) \frac{\partial \tilde{\phi}_k}{\partial z_l}(z) \frac{\partial \psi_{\xi_k}^h}{\partial z_m}(z) |g(\varepsilon_k z)|^{\frac{1}{2}} dz \\
&\quad + \int_{B(0, r/\varepsilon_k)} a(\exp_{\xi_k}(\varepsilon_k z)) \tilde{\phi}_k(z) \psi_{\xi_k}^h(z) |g(\varepsilon_k z)|^{\frac{1}{2}} dz \\
&\quad - \int_{B(0, r/\varepsilon_k)} b(\exp_{\xi_k}(\varepsilon_k z)) f'(V^{\xi_k}(z) \chi(\varepsilon_k z)) \tilde{\phi}_k(z) \psi_{\xi_k}^h(z) |g(\varepsilon_k z)|^{\frac{1}{2}} dz \\
&= \int_{\mathbb{R}^n} c(\xi_k) \nabla \tilde{\phi}_k \nabla \psi_{\xi_k}^h + a(\xi_k) \tilde{\phi}_k \psi_{\xi_k}^h - b(\xi_k) f'(V^{\xi_k}) \tilde{\phi}_k \psi_{\xi_k}^h dz + o(1) = o(1),
\end{aligned}$$

because  $\psi_{\xi_k}^h \rightarrow \psi_{\xi}^h$ ,  $V^{\xi_k} \rightarrow V^{\xi}$  strongly in  $H^1(\mathbb{R}^n)$  and  $\psi_{\xi}^h$  is a weak solution of the linearized equation (2.5). By Proposition 2.1 we have that  $\left\langle Z_{\varepsilon_k, \xi_k}^j, Z_{\varepsilon_k, \xi_k}^h \right\rangle_{\varepsilon_k} = C\delta_{jh} + o(1)$ , where  $C > 0$ . So we conclude that

$$C\alpha_h^k + o(1) = \sum_{j=1}^n \alpha_j^k \left\langle Z_{\varepsilon_k, \xi_k}^j, Z_{\varepsilon_k, \xi_k}^h \right\rangle_{\varepsilon_k} = o(1).$$

This implies that  $\alpha_h^k \rightarrow 0$  for each  $h = 1, \dots, n$ , and, consequently, that  $\|\zeta_k\|_{\varepsilon_k} \rightarrow 0$ .

Setting  $u_k := \phi_k - \psi_k - \zeta_k$ , equation (4.1) can be read as

$$(4.3) \quad -\varepsilon_k^2 \operatorname{div}_g (c(x) \nabla_g u_k) + a(x) u_k = b(x) f'(W_{\varepsilon_k, \xi_k}) u_k + b(x) f'(W_{\varepsilon_k, \xi_k}) (\psi_k + \zeta_k),$$

where  $\|u_k\|_{\varepsilon_k} \rightarrow 1$ . Multiplying (4.3) by  $u_k$  and integrating by parts we get

$$(4.4) \quad \|u_k\|_{\varepsilon_k}^2 = \frac{1}{\varepsilon_k^n} \int_M b(x) (f'(W_{\varepsilon_k, \xi_k}) u_k^2 + f'(W_{\varepsilon_k, \xi_k}) (\psi_k + \zeta_k) u_k).$$

From Holder's inequality, recalling that  $|u|_{\varepsilon, p} \leq C\|u\|_{\varepsilon}$ , we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon_k^n} \int_M b(x) f'(W_{\varepsilon_k, \xi_k}) (\psi_k + \zeta_k) u_k \\
&\leq C \left( \frac{1}{\varepsilon_k^n} \int_M f'(W_{\varepsilon_k, \xi_k})^{\frac{n}{2}} \right)^{\frac{2}{n}} |u_k|_{\varepsilon_k, \frac{2n}{n-2}}^{\frac{n-2}{2n}} |\psi_k + \zeta_k|_{\varepsilon_k, \frac{2n}{n-2}}^{\frac{n-2}{2n}} \\
&\leq C \left( \frac{1}{\varepsilon_k^n} \int_M f'(W_{\varepsilon_k, \xi_k})^{\frac{n}{2}} \right)^{\frac{2}{n}} \|u_k\|_{\varepsilon_k}^{\frac{n-2}{2n}} \|\psi_k + \zeta_k\|_{\varepsilon_k}^{\frac{n-2}{2n}} \\
(4.5) \quad &\leq C \|u_k\|_{\varepsilon_k}^{\frac{n-2}{2n}} \|\psi_k + \zeta_k\|_{\varepsilon_k}^{\frac{n-2}{2n}} = o(1),
\end{aligned}$$

because

$$(4.6) \quad \begin{aligned} \frac{1}{\varepsilon_k^n} \int_M f'(W_{\varepsilon_k, \xi_k})^{\frac{n}{2}} d\mu_g &\leq \frac{1}{\varepsilon_k^n} \int_{B_g(\xi_k, r)} \left( V_{\varepsilon_k}^{\xi_k} \left( \exp_{\xi_k}^{-1}(x) \right) \right)^{\frac{n(p-2)}{2}} d\mu_g \\ &\leq C \int_{B(0, r/\varepsilon)} (V^\xi(z))^{\frac{n(p-2)}{2}} dz \leq C \end{aligned}$$

for some positive constant  $C$ . Combining (4.4), (4.5), (4.6), we get

$$(4.7) \quad \frac{1}{\varepsilon_k^n} \int_M b(x) f'(W_{\varepsilon_k, \xi_k}) u_k^2 \rightarrow 1 \quad \text{as } k \rightarrow +\infty.$$

We will show that this leads to a contradiction. Since  $W_{\varepsilon_k, \xi_k}$  is compactly supported in  $B_g(\xi_k, r)$ , also  $u_k$  is compactly supported in  $B_g(\xi_k, r)$ . Set

$$\tilde{u}_k(z) := u_k(\exp_{\xi_k}(\varepsilon_k z)) \quad \text{for } z \in B(0, r/\varepsilon).$$

We have that

$$\|\tilde{u}_k\|_{H^1(\mathbb{R}^n)} \leq C \|u_k\|_{\varepsilon_k} \leq C,$$

so, up to subsequence, there exists  $\tilde{u} \in H^1(\mathbb{R}^n)$  such that  $\tilde{u}_k \rightarrow \tilde{u}$  weakly in  $H^1(\mathbb{R}^n)$  and strongly in  $L_{\text{loc}}^p(\mathbb{R}^n)$ ,  $p \in (2, 2_n^*)$ .

From (4.3) we deduce that

$$(4.8) \quad -\Delta \tilde{u} + a(\xi) \tilde{u} = b(\xi) f'(V^\xi) \tilde{u}.$$

Indeed, let  $\varphi \in C^\infty(\mathbb{R}^n)$  and let  $\rho > 0$  be such that the support of  $\varphi$  is contained in  $B(0, \rho)$ . For  $k$  large we set

$$\varphi_k(x) := \varphi\left(\frac{1}{\varepsilon_k} \exp_{\xi_k}^{-1}(x)\right) \chi\left(\exp_{\xi_k}^{-1}(x)\right) \quad \text{for } x \in B_g(\xi_k, \varepsilon_k \rho).$$

Thus, by (4.3), we have

$$\begin{aligned} \frac{1}{\varepsilon_k^n} \int_M \varepsilon_k^2 c(x) \nabla_g u_k \nabla_g \varphi_k + a(x) u_k \varphi_k d\mu_g &= \frac{1}{\varepsilon_k^n} \int_M b(x) f'(W_{\varepsilon_k, \xi_k}) u_k \varphi_k d\mu_g \\ &\quad + \frac{1}{\varepsilon_k^n} \int_M b(x) f'(W_{\varepsilon_k, \xi_k}) (\psi_k + \zeta_k) \varphi_k d\mu_g, \end{aligned}$$

where the last term vanishes because  $\psi_k \rightarrow 0$  and  $\zeta_k \rightarrow 0$  in  $H_\varepsilon^1$ . Moreover, we have

$$\begin{aligned} &\frac{1}{\varepsilon_k^n} \int_M \varepsilon_k^2 c(x) \nabla_g u_k \nabla_g \varphi_k \\ &= \int_{B(0, \rho)} \sum_{l, m=1}^n g^{lm}(\varepsilon_k z) c(\exp_{\xi_k}(\varepsilon_k z)) \frac{\partial(\tilde{u}_k(z) \chi(\varepsilon_k z))}{\partial z_l} \frac{\partial \varphi}{\partial z_m} |g(\varepsilon_k z)|^{\frac{1}{2}} dz \\ &= c(\xi_k) \int_{B(0, \rho)} \nabla \tilde{u}(z) \nabla \varphi(z) dz + o(1) \\ &= c(\xi) \int_{B(0, \rho)} \nabla \tilde{u}(z) \nabla \varphi(z) dz + o(1), \end{aligned}$$

$$\begin{aligned}
\frac{1}{\varepsilon_k^n} \int_M a(x) u_k \varphi_k d\mu_g &= \int_{B(0,\rho)} a(\exp_{\xi_k}(\varepsilon_k z)) \tilde{u}_k(z) \varphi(z) |g(\varepsilon_k z)|^{\frac{1}{2}} dz \\
&= a(\xi_k) \int_{B(0,\rho)} \tilde{u}_k(z) \varphi(z) dz + o(1) \\
&= a(\xi) \int_{B(0,\rho)} \tilde{u}_k(z) \varphi(z) dz + o(1)
\end{aligned}$$

and, analogously,

$$\frac{1}{\varepsilon_k^n} \int_M b(x) f'(W_{\varepsilon_k, \xi_k}) u_k \varphi_k d\mu_g = b(\xi) \int_{B(0,\rho)} f'(V^\xi) \tilde{u}_k(z) \varphi(z) dz + o(1),$$

so we get (4.8).

Next we prove that

$$(4.9) \quad \int_{\mathbb{R}_+^n} (c(\xi) \nabla \psi^h \nabla \tilde{u} + a(\xi) \psi^h \tilde{u}) dz = 0 \quad \text{for all } h \in 1, \dots, n-1.$$

In fact, since  $\phi_k, \psi_k \in K_{\varepsilon, \xi}^\perp$  and  $\|\zeta_k\|_{\varepsilon_k} \rightarrow 0$ , we have

$$(4.10) \quad \left| \langle Z_{\varepsilon_k, \xi_k}^h, u_k \rangle_{\varepsilon_k} \right| = \left| \langle Z_{\varepsilon_k, \xi_k}^h, \zeta_k \rangle_{\varepsilon_k} \right| \leq \|Z_{\varepsilon_k, \xi_k}^h\|_{\varepsilon_k} \|\zeta_k\|_{\varepsilon_k} = o(1).$$

On the other hand, using the Taylor expansion of  $g^{ij}(\varepsilon z)$ ,  $|g(\varepsilon z)|^{\frac{1}{2}}$  we get

$$\begin{aligned}
\langle Z_{\varepsilon_k, \xi_k}^h, u_k \rangle_{\varepsilon_k} &= \frac{1}{\varepsilon_k^n} \int_M \varepsilon_k^2 \nabla_g Z_{\varepsilon_k, \xi_k}^h \nabla_g u_k + a(x) Z_{\varepsilon_k, \xi_k}^h u_k \\
&= \int_{B(0, r/\varepsilon_k)} \sum_{l, m=1}^n g^{lm}(\varepsilon_k z) c(\exp_{\xi_k}(\varepsilon z)) \frac{\partial(\psi^h(z) \chi(\varepsilon_k z))}{\partial z_l} \frac{\partial \tilde{u}_k}{\partial z_m} |g(\varepsilon_k z)|^{\frac{1}{2}} dz \\
&\quad + \int_{B(0, r/\varepsilon_k)} a(\exp_{\xi_k}(\varepsilon z)) \psi^h(z) \chi(\varepsilon_k z) \tilde{u}_k |g(\varepsilon_k z)|^{\frac{1}{2}} dz \\
(4.11) \quad &= \int_{\mathbb{R}_+^n} (c(\xi) \nabla \psi^h \nabla \tilde{u} + a(\xi) \psi^h \tilde{u}) dz + o(1),
\end{aligned}$$

because  $c(\exp_{\xi_k}(\varepsilon z)) = c(\xi_k) + O(\varepsilon_k) = c(\xi) + o(1)$  and  $a(\exp_{\xi_k}(\varepsilon z)) = a(\xi) + o(1)$ .

So, from (4.10) and (4.11) we obtain (4.9).

Now, (4.9) and (4.8) imply that  $\tilde{u} = 0$ . Therefore,

$$\begin{aligned}
\frac{1}{\varepsilon_k^n} \int_M f'(W_{\varepsilon_k, \xi_k}(x)) u_k^2(x) d\mu_g &\leq \frac{1}{\varepsilon_k^n} \int_{B_g(0, r)} f'(V_\varepsilon^\xi(\exp_{\xi_k}^{-1}(x))) u_k^2(x) d\mu_g \\
&= C \int_{B(0, r/\varepsilon_k)} f'(V^\xi(z)) \tilde{u}_k^2(z) dz = o(1),
\end{aligned}$$

which contradicts (4.7). This concludes the proof.  $\square$

**Lemma 4.2.** *There exist  $\varepsilon_0 > 0$  and  $C > 0$  such that*

$$\|R_{\varepsilon, \xi}\|_\varepsilon \leq C\varepsilon \text{ for all } \varepsilon \in (0, \varepsilon_0).$$

*Proof.* We recall that, if  $\tilde{v}(\eta) := v(\exp_\xi(\eta))$ , then

$$(4.12) \quad \Delta_g v = \Delta \tilde{v} + (g_\xi^{ij} - \delta_{ij}) \partial_{ij} \tilde{v} - g_\xi^{ij} \Gamma_{ij}^k \partial_k \tilde{v},$$

where  $\Gamma_{ij}^k$  are the Christoffel symbols. Let  $G_{\varepsilon,\xi}$  be a function such that  $W_{\varepsilon,\xi} = i_\varepsilon^*(b(x)G_{\varepsilon,\xi})$ , i.e.

$$-\varepsilon^2 \Delta_g W_{\varepsilon,\xi} + A(x)W_{\varepsilon,\xi} = B(x)G_{\varepsilon,\xi}.$$

Then we have

$$\begin{aligned} B(x)G_{\varepsilon,\xi} &= -\varepsilon^2 \Delta_g W_{\varepsilon,\xi} + A(x)W_{\varepsilon,\xi} \\ &= -\varepsilon^2 \Delta(V_\varepsilon^\xi(\eta)\chi(\eta)) - \varepsilon^2 (g_\xi^{ij} - \delta_{ij}) \partial_{ij}(V_\varepsilon^\xi(\eta)\chi(\eta)) \\ &\quad + \varepsilon^2 g_\xi^{ij} \Gamma_{ij}^k \partial_k(V_\varepsilon^\xi(\eta)\chi(\eta)) + A(\exp_\xi(\eta))V_\varepsilon^\xi(\eta)\chi(\eta) \\ &= -\varepsilon^2 V_\varepsilon^\xi(\eta) \Delta \chi(\eta) - 2\varepsilon^2 \nabla V_\varepsilon^\xi(\eta) \nabla \chi(\eta) - \varepsilon^2 (g_\xi^{ij} - \delta_{ij}) \partial_{ij}(V_\varepsilon^\xi(\eta)\chi(\eta)) \\ &\quad + \varepsilon^2 g_\xi^{ij} \Gamma_{ij}^k \partial_k(V_\varepsilon^\xi(\eta)\chi(\eta)) + [A(\exp_\xi(\eta)) - A(\xi)]V_\varepsilon^\xi(\eta)\chi(\eta) \\ &\quad + B(\xi)V_\varepsilon^\xi(\eta)\chi(\eta) \end{aligned}$$

using (2.2). By definition of  $R_{\varepsilon,\xi}$  and inequality (3.1) we have that

$$\|R_{\varepsilon,\xi}\|_\varepsilon \leq \|i_\varepsilon^*(b(x)f(W_{\varepsilon,\xi})) - W_{\varepsilon,\xi}\|_\varepsilon \leq |W_{\varepsilon,\xi}^{p-1} - G_{\varepsilon,\xi}|_{p',\varepsilon}.$$

Now

$$\begin{aligned} \int_M |W_{\varepsilon,\xi}^{p-1} - G_{\varepsilon,\xi}|^{p'} d\mu_g &\leq C \int_{B(0,r)} \left| (V_\varepsilon^\xi(\eta)\chi(\eta))^{p-1} - G_{\varepsilon,\xi}(\exp_\xi(\eta)) \right|^{p'} d\eta \\ &\leq C \int_{B(0,r)} \left| (V_\varepsilon^\xi(\eta))^{p-1} [\chi(\eta)^{p-1} - \chi(\eta)] \right|^{p'} d\eta \\ &\quad + C\varepsilon^{2p'} \int_{B(0,r)} |V_\varepsilon^\xi(\eta) \Delta \chi(\eta)|^{p'} d\eta \\ &\quad + C\varepsilon^{2p'} \int_{B(0,r)} |\nabla V_\varepsilon^\xi(\eta) \nabla \chi(\eta)|^{p'} d\eta \\ &\quad + C\varepsilon^{2p'} \int_{B(0,r)} \left| (g_\xi^{ij}(\eta) - \delta_{ij}) \partial_{ij}(V_\varepsilon^\xi(\eta)\chi(\eta)) \right|^{p'} d\eta \\ &\quad + C\varepsilon^{2p'} \int_{B(0,r)} \left| g_\xi^{ij}(\eta) \Gamma_{ij}^k(\eta) \partial_k(V_\varepsilon^\xi(\eta)\chi(\eta)) \right|^{p'} d\eta \\ &\quad + C \int_{B(0,r)} |[A(\exp_\xi(\eta)) - a(\xi)]V_\varepsilon^\xi(\eta)\chi(\eta)|^{p'} d\eta \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned}$$

Using the exponential decay of  $V_\varepsilon^\xi$  and its derivative we get

$$I_1 + I_2 + I_3 = o(\varepsilon^{n+2p'}).$$

Moreover,

$$\begin{aligned} I_4 &= C\varepsilon^{2p'} \int_{B(0,r)} \left| (g_\xi^{ij}(\eta) - \delta_{ij}) \partial_{ij}(V_\varepsilon^\xi(\eta)\chi(\eta)) \right|^{p'} d\eta \\ &= C\varepsilon^{2p'} \int_{B(0,r)} \left| (g_\xi^{ij}(\eta) - \delta_{ij}) [\partial_{ij} V_\varepsilon^\xi(\eta)] \chi(\eta) \right|^{p'} d\eta + o(\varepsilon^{n+2p'}) \\ &= C\varepsilon^n \int_{B(0,r/\varepsilon)} \left| (g_\xi^{ij}(\varepsilon z) - \delta_{ij}) [\partial_{ij} V_\varepsilon^\xi(z)] \chi(\varepsilon z) \right|^{p'} dz + o(\varepsilon^{n+2p'}) \\ &= O(\varepsilon^{n+2p'}) \end{aligned}$$

in light of the Taylor expansion of  $g_\xi^{ij}(\varepsilon z)$ , and

$$\begin{aligned}
I_5 &= C\varepsilon^{2p'} \int_{B(0,r)} \left| g_\xi^{ij}(\eta) \Gamma_{ij}^k(\eta) \partial_k (V_\varepsilon^\xi(\eta) \chi(\eta)) \right|^{p'} d\eta \\
&= C\varepsilon^{2p'} \int_{B(0,r)} \left| \Gamma_{ij}^k(\eta) [\partial_k V_\varepsilon^\xi(\eta)] \chi(\eta) \right|^{p'} d\eta + o(\varepsilon^{n+2p'}) \\
&= C\varepsilon^{p'+N} \int_{B(0,r/\varepsilon)} \left| \Gamma_{ij}^k(\varepsilon z) [\partial_k V^\xi(z)] \chi(\varepsilon z) \right|^{p'} dz + o(\varepsilon^{n+2p'}) \\
&= O(\varepsilon^{n+2p'})
\end{aligned}$$

since  $\Gamma_{ij}^k(\varepsilon z) = \Gamma_{ij}^k(0) + O(\varepsilon|z|) = O(\varepsilon|z|)$ . Finally, setting  $\tilde{A}(\eta) = A(\exp_\xi(\eta))$  and using the Taylor expansion of  $\tilde{A}(\varepsilon z)$ , we get

$$\begin{aligned}
I_6 &= C \int_{B(0,r)} \left| [\tilde{A}(\eta) - \tilde{A}(0)] V_\varepsilon^\xi(\eta) \chi(\eta) \right|^{p'} d\eta = \\
&= C\varepsilon^n \int_{B(0,r/\varepsilon)} \left| [\tilde{A}(\varepsilon z) - \tilde{A}(0)] V^\xi(z) \chi(\varepsilon z) \right|^{p'} dz = O(\varepsilon^{n+p'}).
\end{aligned}$$

This concludes the proof.  $\square$

*Proof of Proposition 3.1.* We use a fixed point argument to show existence of a solution to equation (3.3). We define the operator

$$\begin{aligned}
T_{\varepsilon,\xi} &: K_{\varepsilon,\xi}^\perp \rightarrow K_{\varepsilon,\xi}^\perp \\
T_{\varepsilon,\xi}(\phi) &:= L_{\varepsilon,\xi}^{-1} (N_{\varepsilon,\xi}(\phi) + R_{\varepsilon,\xi})
\end{aligned}$$

By Lemma 4.1,  $T_{\varepsilon,\xi}$  is well defined and the inequalities

$$\begin{aligned}
\|T_{\varepsilon,\xi}(\phi)\|_\varepsilon &\leq C (\|N_{\varepsilon,\xi}(\phi)\|_\varepsilon + \|R_{\varepsilon,\xi}\|_\varepsilon) \\
\|T_{\varepsilon,\xi}(\phi_1) - T_{\varepsilon,\xi}(\phi_2)\|_\varepsilon &\leq C (\|N_{\varepsilon,\xi}(\phi_1) - N_{\varepsilon,\xi}(\phi_2)\|_\varepsilon)
\end{aligned}$$

hold true for some suitable constant  $C > 0$ . From the mean value theorem and inequality (3.1) we get

$$\|N_{\varepsilon,\xi}(\phi_1) - N_{\varepsilon,\xi}(\phi_2)\|_\varepsilon \leq C |f'(W_{\varepsilon,\xi} + \phi_2 + t(\phi_1 - \phi_2)) - f'(W_{\varepsilon,\xi})|_{\frac{p}{p-2}, \varepsilon} \|\phi_1 - \phi_2\|_\varepsilon.$$

Using (A.2) we can prove that  $|f'(W_{\varepsilon,\xi} + \phi_2 + t(\phi_1 - \phi_2)) - f'(W_{\varepsilon,\xi})|_{\frac{p}{p-2}, \varepsilon} < 1$  provided  $\|\phi_1\|_\varepsilon$  and  $\|\phi_2\|_\varepsilon$  small enough. Thus, there exists  $0 < C < 1$  such that  $\|T_{\varepsilon,\xi}(\phi_1) - T_{\varepsilon,\xi}(\phi_2)\|_\varepsilon \leq C \|\phi_1 - \phi_2\|_\varepsilon$ . Moreover, from the same estimates we get

$$\|N_{\varepsilon,\xi}(\phi)\|_\varepsilon \leq C (\|\phi\|_\varepsilon^2 + \|\phi\|_\varepsilon^{p-1}).$$

This, combined with Lemma 4.2, gives

$$\|T_{\varepsilon,\xi}(\phi)\|_\varepsilon \leq C (\|N_{\varepsilon,\xi}(\phi)\|_\varepsilon + \|R_{\varepsilon,\xi}\|_\varepsilon) \leq C (\|\phi\|_\varepsilon^2 + \|\phi\|_\varepsilon^{p-1} + \varepsilon).$$

So, there exists  $C > 0$  such that  $T_{\varepsilon,\xi}$  maps the ball of center 0 and radius  $C\varepsilon$  in  $K_{\varepsilon,\xi}^\perp$  into itself, and it is a contraction. It follows that  $T_{\varepsilon,\xi}$  has a fixed point  $\phi_{\varepsilon,\xi}$  with norm  $\|\phi_{\varepsilon,\xi}\|_\varepsilon \leq \varepsilon$ .

The continuity of  $\phi_{\varepsilon,\xi}$  with respect to  $\xi$  can be proved by similar arguments via the implicit function theorem.  $\square$

## 5. THE REDUCED FUNCTIONAL

In this section we obtain the expansion of the reduced energy functional  $\tilde{J}_\varepsilon(\xi) := J_\varepsilon(W_{\varepsilon,\xi} + \Phi_{\varepsilon,\xi})$  stated in Proposition 3.3.

**Lemma 5.1.** *We have that*

$$\tilde{J}_\varepsilon(\xi) := J_\varepsilon(W_{\varepsilon,\xi} + \Phi_{\varepsilon,\xi}) = J_\varepsilon(W_{\varepsilon,\xi}) + O(\varepsilon)$$

$\mathcal{C}^1$ -uniformly with respect to  $\xi \in M$  as  $\varepsilon \rightarrow 0$ .

*Proof.* We divide the proof into two steps: first, we show that the estimate holds true  $\mathcal{C}^0$ -uniformly with respect to  $\xi$ , and then we show that it holds true  $\mathcal{C}^1$ -uniformly as well.

**Step 1:**  $J_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) = J_\varepsilon(W_{\varepsilon,\xi}) + O(\varepsilon)$  holds true  $\mathcal{C}^0$ -uniformly with respect to  $\xi \in M$  as  $\varepsilon \rightarrow 0$ .

Indeed, by (3.3), we have that

$$\begin{aligned} J_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - J_\varepsilon(W_{\varepsilon,\xi}) &= \frac{1}{2} \|\phi_{\varepsilon,\xi}\|_\varepsilon^2 + \frac{1}{\varepsilon^n} \int \varepsilon^2 c(x) \nabla W_{\varepsilon,\xi} \nabla \phi_{\varepsilon,\xi} + a(x) W_{\varepsilon,\xi} \phi_{\varepsilon,\xi} \\ &\quad - \frac{1}{\varepsilon^n} \int b(x) [F(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - F(W_{\varepsilon,\xi})] \\ &= -\frac{1}{2} \|\phi_{\varepsilon,\xi}\|_\varepsilon^2 + \frac{1}{\varepsilon^n} \int b(x) [f(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - f(W_{\varepsilon,\xi})] \phi_{\varepsilon,\xi} \\ &\quad - \frac{1}{\varepsilon^n} \int b(x) [F(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - F(W_{\varepsilon,\xi}) - f(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) \phi_{\varepsilon,\xi}], \end{aligned}$$

where  $F(u) = \frac{1}{p} |u^+|^p$ . Using the mean value theorem we obtain

$$\begin{aligned} \int b(x) [f(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - f(W_{\varepsilon,\xi})] \phi_{\varepsilon,\xi} &= \int b(x) f'(W_{\varepsilon,\xi} + t_1 \phi_{\varepsilon,\xi}) \phi_{\varepsilon,\xi}^2 \\ \int b(x) [F(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - F(W_{\varepsilon,\xi}) - f(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) \phi_{\varepsilon,\xi}] &= \int b(x) f'(W_{\varepsilon,\xi} + t_2 \phi_{\varepsilon,\xi}) \phi_{\varepsilon,\xi}^2 \end{aligned}$$

for some  $t_1, t_2 \in (0, 1)$ . Now, since  $\|\phi_{\varepsilon,\xi}\|_\varepsilon = O(\varepsilon)$ ,

$$\begin{aligned} \frac{1}{\varepsilon^n} \int |b(x) f'(W_{\varepsilon,\xi} + t_1 \phi_{\varepsilon,\xi}) \phi_{\varepsilon,\xi}^2| &\leq \frac{C}{\varepsilon^n} \int W_{\varepsilon,\xi}^{p-2} \phi_{\varepsilon,\xi}^2 + \frac{C}{\varepsilon^n} \int \phi_{\varepsilon,\xi}^p \\ &\leq |\phi_{\varepsilon,\xi}|_{2,\varepsilon}^2 + |\phi_{\varepsilon,\xi}|_{p,\varepsilon}^p \\ &\leq \|\phi_{\varepsilon,\xi}\|_\varepsilon^2 + \|\phi_{\varepsilon,\xi}\|_\varepsilon^p = o(\varepsilon) \end{aligned}$$

and the claim follows.

**Step 2:** Setting  $\xi(y) = \exp_{\xi_0}(y)$ , we have that

$$\left. \frac{\partial}{\partial y_h} J_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) \right|_{y=0} = \left. \frac{\partial}{\partial y_h} J_\varepsilon(W_{\varepsilon,\xi(y)}) \right|_{y=0} + O(\varepsilon)$$

$\mathcal{C}^0$ -uniformly with respect to  $\xi \in M$  as  $\varepsilon \rightarrow 0$  for all  $h = 1, \dots, n$ .

Since the proof of this statement is lengthy, we postpone it to Appendix A.  $\square$

Now we expand the function  $\xi \mapsto J_\varepsilon(W_{\varepsilon,\xi})$ .



**Lemma 5.2.** *The expansion*

$$\begin{aligned} J_\varepsilon(W_{\varepsilon,\xi}) &= \frac{1}{2} \int_{\mathbb{R}^n} c(\xi) |\nabla V^\xi(z)|^2 + a(\xi) (V^\xi)^2(z) dz - \frac{1}{p} \int_{\mathbb{R}^n} b(\xi) (V^\xi)^p(z) dz + O(\varepsilon) \\ &= \left( \frac{p-2}{2p} \int_{\mathbb{R}^n} U^p dz \right) \frac{c(\xi)^{\frac{n}{2}} a(\xi)^{\frac{p}{p-2} - \frac{n}{2}}}{b(\xi)^{\frac{2}{p-2}}} + O(\varepsilon). \end{aligned}$$

holds true  $\mathcal{C}^1$ -uniformly with respect to  $\xi \in M$ .

*Proof.* We perform the proof in three steps.

**Step1:** The equality

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}^n} c(\xi) |\nabla V^\xi(z)|^2 + a(\xi) (V^\xi)^2(z) dz - \frac{1}{p} \int_{\mathbb{R}^n} b(\xi) (V^\xi)^p(z) dz \\ &= \left( \frac{p-2}{2p} \int_{\mathbb{R}^n} U^p dz \right) \frac{c(\xi)^{\frac{n}{2}} a(\xi)^{\frac{p}{p-2} - \frac{n}{2}}}{b(\xi)^{\frac{2}{p-2}}} \end{aligned}$$

holds true for every  $\xi \in M$ .

This follows by straightforward computation using (2.3) and (2.4).

**Step2:** The estimate

$$J_\varepsilon(W_{\varepsilon,\xi}) = \frac{1}{2} \int_{\mathbb{R}^n} c(\xi) |\nabla V^\xi(z)|^2 + a(\xi) (V^\xi)^2(z) dz - \frac{1}{p} \int_{\mathbb{R}^n} b(\xi) (V^\xi)^p(z) dz + O(\varepsilon)$$

holds true  $\mathcal{C}^0$ -uniformly with respect to  $\xi \in M$  as  $\varepsilon \rightarrow 0$ .

Indeed, in normal coordinates we have

$$\begin{aligned} J_\varepsilon(W_{\varepsilon,\xi}) &= \frac{1}{2} \int_{|z| < \frac{r}{\varepsilon}} \sum_{i,j=1}^n g^{ij}(\varepsilon z) \tilde{c}(\varepsilon z) \frac{\partial (V^\xi(z) \chi_r(\varepsilon z))}{\partial z_i} \frac{\partial (V^\xi(z) \chi_r(\varepsilon z))}{\partial z_j} |g(\varepsilon z)|^{\frac{1}{2}} dz \\ &\quad + \frac{1}{2} \int_{|z| < \frac{r}{\varepsilon}} \tilde{a}(\varepsilon z) (V^\xi(z) \chi_r(\varepsilon z))^2 |g(\varepsilon z)|^{\frac{1}{2}} dz \\ &\quad - \frac{1}{p} \int_{|z| < \frac{r}{\varepsilon}} \tilde{b}(\varepsilon z) (V^\xi(z) \chi_r(\varepsilon z))^p |g(\varepsilon z)|^{\frac{1}{2}} dz, \end{aligned}$$

where  $\varepsilon z := \exp_\xi^{-1} x$ , and  $\tilde{c}(\varepsilon z) := c(x)$ ,  $\tilde{a}(\varepsilon z) := a(x)$  and  $\tilde{b}(\varepsilon z) := b(x)$ . Using the expansions (2.8) and (2.9) and collecting terms of the same order, we get

$$J_\varepsilon(W_{\varepsilon,\xi}) = \frac{1}{2} \int_{\mathbb{R}^n} c(\xi) |\nabla V^\xi(z)|^2 + a(\xi) (V^\xi)^2(z) dz - \frac{1}{p} \int_{\mathbb{R}^n} b(\xi) (V^\xi)^p(z) dz + O(\varepsilon),$$

as claimed.

**Step 3:** The estimate

$$\begin{aligned} &\left. \frac{\partial}{\partial y_h} J_\varepsilon(W_{\varepsilon,\xi(y)}) \right|_{y=0} = \\ &= \left. \frac{\partial}{\partial y_h} \left( \frac{1}{2} \int_{\mathbb{R}^n} c(\xi(y)) |\nabla V^{\xi(y)}|^2 + a(\xi(y)) (V^{\xi(y)})^2 dz - \frac{1}{p} \int_{\mathbb{R}^n} b(\xi(y)) (V^{\xi(y)})^p(z) dz \right) \right|_{y=0} + O(\varepsilon) \end{aligned}$$

holds true  $\mathcal{C}^0$ -uniformly with respect to  $\xi \in M$  as  $\varepsilon \rightarrow 0$  for all  $h = 1, \dots, n$ . Here  $\xi(y) := \exp_{\xi_0}(y)$  with  $y \in B(0, r)$ .

The proof of this statement is technical and we postpone it to Appendix A  $\square$

## APPENDIX A. SOME TECHNICAL FACTS

Here we collect some technical facts we have used to prove some of the results, and we give the missing proofs.

The proofs of Lemmas A.1, A.2 and A.4 are similar to those of Lemmas 6.1, 6.2 and 6.3 of [22] and will just be sketched.

**Lemma A.1.** *The following estimates hold true*

$$\left\| \frac{\partial}{\partial y_h} Z_{\varepsilon, \xi(y)}^l \right\|_{\varepsilon} = O\left(\frac{1}{\varepsilon}\right), \quad \left\| \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(y)} \right\|_{\varepsilon} = O\left(\frac{1}{\varepsilon}\right).$$

*Proof.* The proof follows by direct computation.  $\square$

**Lemma A.2.** *The following estimates hold true*

$$\left\langle Z_{\varepsilon, \xi_0}^l, \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(y)} \Big|_{y=0} \right\rangle_{\varepsilon} = -\frac{1}{\varepsilon} C \delta_{hl} + o\left(\frac{1}{\varepsilon}\right),$$

where  $C \in \mathbb{R}$  is a suitable constant.

*Proof.* The proof follows from the definitions of  $Z_{\varepsilon, \xi_0}^l$  and  $W_{\varepsilon, \xi(y)}$  and the Taylor expansion of  $\frac{\partial \mathcal{E}_k}{\partial y_h}$ .  $\square$

**Lemma A.3.** *There exists  $C > 0$  such that, for  $\varepsilon$  small enough,*

$$\|Z_{\varepsilon, \xi}^h - i_{\varepsilon}^*[b(x)f'(W_{\varepsilon, \xi(y)})Z_{\varepsilon, \xi}^h]\|_{\varepsilon} \leq C\varepsilon$$

for all  $\xi \in M$ ,  $h = 1, \dots, n$ .

*Proof.* Let  $G_{\varepsilon, \xi}$  be a function such that  $Z_{\varepsilon, \xi}^h(x) = i_{\varepsilon}^*(b(x)G_{\varepsilon, \xi})$ , i.e.

$$-\varepsilon^2 \Delta_g Z_{\varepsilon, \xi}^h + A(x)Z_{\varepsilon, \xi}^h = B(x)G_{\varepsilon, \xi}.$$

Thus, using (4.12) we have

$$\begin{aligned} b(x)G_{\varepsilon, \xi} &= -\varepsilon^2 \Delta_g Z_{\varepsilon, \xi} + A(x)Z_{\varepsilon, \xi} \\ &= -\varepsilon^2 \psi_{\xi}^h(\eta/\varepsilon) \Delta \chi(\eta) - 2\varepsilon^2 \nabla \psi_{\xi}^h(\eta/\varepsilon) \nabla \chi(\eta) - \varepsilon^2 (g_{\xi}^{ij} - \delta_{ij}) \partial_{ij}(\psi_{\xi}^h(\eta/\varepsilon) \chi(\eta)) \\ &\quad + \varepsilon^2 g_{\xi}^{ij} \Gamma_{ij}^k \partial_k(\psi_{\xi}^h(\eta/\varepsilon) \chi(\eta)) + [A(\exp_{\xi}(\eta)) - a(\xi)] \psi_{\xi}^h(\eta/\varepsilon) \chi(\eta) \\ &\quad + (p-1)B(\xi) (V_{\varepsilon}^{\xi}(\eta))^{p-2} \psi_{\xi}^h(\eta/\varepsilon) \chi(\eta), \end{aligned}$$

by (2.5). Now the proof follows as in Lemma 4.2.  $\square$

**Lemma A.4.** *There exists  $C > 0$  such that, for  $\varepsilon$  small enough,*

$$\left\| \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(y)} + \frac{1}{\varepsilon} Z_{\varepsilon, \xi(y)}^h \right\|_{\varepsilon} \leq C\varepsilon$$

for all  $\xi_0 \in M$ ,  $h = 1, \dots, n$ .

*Proof.* The proof follows from the definitions of  $Z_{\varepsilon, \xi_0}^l$  and  $W_{\varepsilon, \xi(y)}$  and the Taylor expansion of  $\frac{\partial \mathcal{E}_k}{\partial y_h}$ .  $\square$

*Proof of Lemma 5.1, Step 2.* We have

$$\begin{aligned}
\left. \frac{\partial}{\partial y_h} J_\varepsilon(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) - J_\varepsilon(W_{\varepsilon,\xi(y)}) \right|_{y=0} &= J'_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) \left[ \left. \frac{\partial}{\partial y_h} (W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) \right|_{y=0} \right] \\
&\quad - J'_\varepsilon(W_{\varepsilon,\xi}) \left[ \left. \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right|_{y=0} \right] \\
&= (J'_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - J'_\varepsilon(W_{\varepsilon,\xi})) \left[ \left. \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right|_{y=0} \right] \\
&\quad + J'_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) \left[ \left. \frac{\partial}{\partial y_h} \phi_{\varepsilon,\xi(y)} \right|_{y=0} \right].
\end{aligned}$$

In light of (3.3), the last term is

$$J'_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) \left[ \left. \frac{\partial}{\partial y_h} \phi_{\varepsilon,\xi(y)} \right|_{y=0} \right] = \sum_{l=i}^n C_\varepsilon^l \left\langle Z_{\varepsilon,\xi}^l, \left. \frac{\partial}{\partial y_h} \phi_{\varepsilon,\xi(y)} \right|_{y=0} \right\rangle_\varepsilon.$$

Since  $\phi_{\varepsilon,\xi} \in K_{\varepsilon,\xi}^\perp$ , we have

$$\left\langle Z_{\varepsilon,\xi(y)}^l, \left. \frac{\partial}{\partial y_h} \phi_{\varepsilon,\xi(y)} \right|_\varepsilon \right\rangle = - \left\langle \left. \frac{\partial}{\partial y_h} Z_{\varepsilon,\xi(y)}^l, \phi_{\varepsilon,\xi(y)} \right\rangle_\varepsilon = O \left( \left\| \left. \frac{\partial}{\partial y_h} Z_{\varepsilon,\xi(y)}^l \right\|_\varepsilon \cdot \phi_{\varepsilon,\xi(y)} \right) = O(1)$$

by Lemma A.1. Moreover, we claim that

$$(A.1) \quad \sum_{l=1}^n |C_\varepsilon^l| = O(\varepsilon).$$

Indeed, since  $\phi_{\varepsilon,\xi} \in K_{\varepsilon,\xi}^\perp$ ,

$$\begin{aligned}
J'_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi})[Z_{\varepsilon,\xi}^l] &= \langle W_{\varepsilon,\xi}, Z_{\varepsilon,\xi}^l \rangle_\varepsilon - \frac{1}{\varepsilon^n} \int b(x) f(W_{\varepsilon,\xi}) Z_{\varepsilon,\xi}^l \\
&\quad + \frac{1}{\varepsilon^n} \int b(x) (f(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - f(W_{\varepsilon,\xi})) Z_{\varepsilon,\xi}^l.
\end{aligned}$$

Now, from the exponential decay of  $V^\xi$  and its derivatives, and the expansion of  $g^{ij}$  and  $|g|$ , we get

$$\begin{aligned}
&\langle W_{\varepsilon,\xi}, Z_{\varepsilon,\xi}^l \rangle_\varepsilon - \frac{1}{\varepsilon^n} \int b(x) f(W_{\varepsilon,\xi}) Z_{\varepsilon,\xi}^l \\
&= \frac{1}{2} \int_{\mathbb{R}^n} c(x) \nabla V^\xi \nabla \psi_\xi^l + a(x) V^\xi \psi_\xi^l - \frac{1}{p} \int_{\mathbb{R}^n} b(x) (V^\xi)^{p-1} \psi_\xi^l + O(\varepsilon^2) = O(\varepsilon^2)
\end{aligned}$$

and, by the mean value theorem, for some  $t \in (0, 1)$  we have that

$$\begin{aligned}
\frac{1}{\varepsilon^n} \left| \int (f(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - f(W_{\varepsilon,\xi})) Z_{\varepsilon,\xi}^l \right| &= \frac{1}{\varepsilon^n} \left| \int f'(W_{\varepsilon,\xi} + t\phi_{\varepsilon,\xi}) \phi_{\varepsilon,\xi} Z_{\varepsilon,\xi}^l \right| \\
&\leq C |W_{\varepsilon,\xi}|_{p,\varepsilon}^{p-2} |\phi_{\varepsilon,\xi}|_{p,\varepsilon} |Z_{\varepsilon,\xi}^l|_{p,\varepsilon} + C |\phi_{\varepsilon,\xi}|_{p,\varepsilon}^{p-1} |Z_{\varepsilon,\xi}^l|_{p,\varepsilon} = O(\varepsilon),
\end{aligned}$$

Thus, (A.1) holds true and

$$J'_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) \left[ \left. \frac{\partial}{\partial y_h} \phi_{\varepsilon,\xi(y)} \right|_{y=0} \right] = O(\varepsilon).$$

We prove now that

$$(J'_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - J'_\varepsilon(W_{\varepsilon,\xi})) \left[ \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \Big|_{y=0} \right] = O(\varepsilon).$$

We have

$$\begin{aligned} & (J'_\varepsilon(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) - J'_\varepsilon(W_{\varepsilon,\xi(y)})) \left[ \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right] \\ &= \left\langle \phi_{\varepsilon,\xi(y)}, \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right\rangle_\varepsilon - \frac{1}{\varepsilon^n} \int b(x) (f(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) - f(W_{\varepsilon,\xi(y)})) \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \\ &= \left\langle \phi_{\varepsilon,\xi(y)} - i_\varepsilon^*[b(x)f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)}], \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right\rangle_\varepsilon \\ &\quad - \frac{1}{\varepsilon^n} \int b(x) (f(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) - f(W_{\varepsilon,\xi(y)}) - f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)}) \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \\ &= \left\langle \phi_{\varepsilon,\xi(y)} - i_\varepsilon^*[b(x)f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)}], \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} + \frac{1}{\varepsilon} Z_{\varepsilon,\xi(y)}^h \right\rangle_\varepsilon \\ &\quad - \frac{1}{\varepsilon} \left\langle \phi_{\varepsilon,\xi(y)}, Z_{\varepsilon,\xi(y)}^h - i_\varepsilon^*[b(x)f'(W_{\varepsilon,\xi(y)})Z_{\varepsilon,\xi(y)}^h] \right\rangle_\varepsilon \\ &\quad - \frac{1}{\varepsilon^n} \int b(x) (f(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) - f(W_{\varepsilon,\xi(y)}) - f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)}) \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

By Lemma A.3 we have that

$$|I_2| \leq \frac{1}{\varepsilon} \|Z_{\varepsilon,\xi}^h - i_\varepsilon^*[b(x)f'(W_{\varepsilon,\xi(y)})Z_{\varepsilon,\xi}^h]\|_\varepsilon \|\phi_{\varepsilon,\xi(y)}\|_\varepsilon = O(\varepsilon).$$

In order to estimate  $I_3$  we observe that

$$(A.2) \quad |f'(W_{\varepsilon,\xi} + v) - f'(W_{\varepsilon,\xi})| \leq \begin{cases} CW_{\varepsilon,\xi}^{p-3}|v| & 2 < p < 3, \\ C(W_{\varepsilon,\xi}^{p-3}|v| + |v|^{p-2}) & p \geq 3. \end{cases}$$

We prove that  $I_3 = O(\varepsilon)$  only for  $p \geq 3$ ; the other case is similar. By mean value theorem, (A.2) and Lemma A.1 we have that

$$\begin{aligned} |I_3| &\leq \frac{C}{\varepsilon^n} \int \left( W_{\varepsilon,\xi(y)}^{p-3} \phi_{\varepsilon,\xi(y)}^2 + |\phi_{\varepsilon,\xi(y)}|^{p-1} \right) \left| \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right| \\ &\leq (|W_{\varepsilon,\xi(y)}|_{p,\varepsilon}^{p-3} |\phi_{\varepsilon,\xi(y)}|_{2,\varepsilon}^2 + |\phi_{\varepsilon,\xi(y)}|_{p,\varepsilon}^{p-1}) \left| \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right|_{p,\varepsilon} \\ &\leq (\|\phi_{\varepsilon,\xi(y)}\|_\varepsilon^2 + \|\phi_{\varepsilon,\xi(y)}\|_\varepsilon^{p-1}) \left\| \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right\|_\varepsilon = O(\varepsilon) \end{aligned}$$

Finally, for  $I_1$  we have that

$$|I_1| \leq \|\phi_{\varepsilon,\xi(y)} - i_\varepsilon^*[b(x)f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)}]\|_\varepsilon \left\| \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} + \frac{1}{\varepsilon} Z_{\varepsilon,\xi(y)}^h \right\|_\varepsilon \leq O(\varepsilon),$$

because

$$\begin{aligned} \|\phi_{\varepsilon,\xi(y)} - i_\varepsilon^*[b(x)f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)}]\|_\varepsilon &\leq \|\phi_{\varepsilon,\xi(y)}\|_\varepsilon + \|i_\varepsilon^*[b(x)f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)}]\|_\varepsilon \\ &\leq \|\phi_{\varepsilon,\xi(y)}\|_\varepsilon + C|f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)}|_{p',\varepsilon} \\ &\leq \|\phi_{\varepsilon,\xi(y)}\|_\varepsilon + C|W_{\varepsilon,\xi(y)}|_{p,\varepsilon}^{p-2} |\phi_{\varepsilon,\xi(y)}|_{p,\varepsilon} = O(\varepsilon) \end{aligned}$$

and, by Lemma A.4,

$$\left\| \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(y)} + \frac{1}{\varepsilon} Z_{\varepsilon, \xi(y)}^h \right\|_{\varepsilon} \leq C\varepsilon.$$

This concludes the proof.  $\square$

*Proof of Lemma 5.2, Step 3.* We prove the claim for  $h = 1$ . By (2.6) we have

$$\begin{aligned} \left. \frac{\partial}{\partial y_1} J_{\varepsilon}(W_{\varepsilon, \xi(y)}) \right|_{y=0} &= J'_{\varepsilon}(W_{\varepsilon, \xi(y)}) \left[ \frac{\partial}{\partial y_1} W_{\varepsilon, \xi(y)} \right] \Big|_{y=0} \\ &= \frac{\varepsilon^2}{\varepsilon^n} \int_M c(x) \nabla_g W_{\varepsilon, \xi(y)} \nabla_g \frac{\partial}{\partial y_1} W_{\varepsilon, \xi(y)} d\mu_g \Big|_{y=0} \\ &\quad + \frac{1}{\varepsilon^n} \int_M a(x) W_{\varepsilon, \xi(y)} \frac{\partial}{\partial y_1} W_{\varepsilon, \xi(y)} d\mu_g \Big|_{y=0} \\ &\quad + \frac{1}{\varepsilon^n} \int_M b(x) W_{\varepsilon, \xi(y)}^{p-1} \frac{\partial}{\partial y_1} W_{\varepsilon, \xi(y)} d\mu_g \Big|_{y=0} \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

To simplify the notation, set  $x := \exp_{\xi_0}(\eta)$  and

$$\tilde{\mathcal{E}}(y, \eta) := \exp_{\xi(y)}^{-1} \exp_{\xi_0}(\eta) = \exp_{\xi(y)}^{-1}(x) = \mathcal{E}(y, x),$$

so

$$\tilde{\mathcal{E}}(0, \varepsilon z) = \exp_{\xi_0}^{-1} \exp_{\xi_0}(\varepsilon z) = \varepsilon z.$$

For the term  $I_1$  we have

$$\begin{aligned} I_1 &= \frac{\varepsilon^2}{\varepsilon^n} \int_M c(x) \nabla_g W_{\varepsilon, \xi(y)} \nabla_g \frac{\partial}{\partial y_1} W_{\varepsilon, \xi(y)} d\mu_g \Big|_{y=0} \\ &= \frac{\varepsilon^2}{\varepsilon^n} \int_{\mathbb{R}^n} c(\exp_{\xi_0} \eta) |g_{\xi_0}(\eta)|^{\frac{1}{2}} g_{\xi_0}^{ij}(\eta) \frac{\partial}{\partial \eta_i} \left[ \tilde{\gamma}(0) U_{\varepsilon}(\sqrt{\tilde{A}(0)} \tilde{\mathcal{E}}(0, \eta)) \chi(\tilde{\mathcal{E}}(0, \eta)) \right] \\ &\quad \times \frac{\partial}{\partial \eta_j} \frac{\partial}{\partial y_1} \left[ \tilde{\gamma}(y) U_{\varepsilon}(\sqrt{\tilde{A}(y)} \tilde{\mathcal{E}}(y, \eta)) \chi(\tilde{\mathcal{E}}(y, \eta)) \right] \Big|_{y=0} d\eta \\ &= \int_{\mathbb{R}^n} \tilde{c}(\varepsilon z) |g_{\xi_0}(\varepsilon z)|^{\frac{1}{2}} g_{\xi_0}^{ij}(\varepsilon z) \frac{\partial}{\partial z_i} \left[ \tilde{\gamma}(0) U_{\varepsilon}(\sqrt{\tilde{A}(0)} \tilde{\mathcal{E}}(0, \varepsilon z)) \chi(\tilde{\mathcal{E}}(0, \varepsilon z)) \right] \\ &\quad \times \frac{\partial}{\partial z_j} \frac{\partial}{\partial y_1} \left[ \tilde{\gamma}(y) U_{\varepsilon}(\sqrt{\tilde{A}(y)} \tilde{\mathcal{E}}(y, \varepsilon z)) \chi(\tilde{\mathcal{E}}(y, \varepsilon z)) \right] \Big|_{y=0} dz. \end{aligned}$$

Now, recalling that  $\tilde{\mathcal{E}}(0, \varepsilon z) = \exp_{\xi_0}^{-1} \exp_{\xi_0}(\varepsilon z) = \varepsilon z$ , from equations (2.6), (2.7), (2.8), (2.9) and the exponential decay of  $U$  and its derivatives, we get

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^n} \tilde{c}(\varepsilon z) \tilde{\gamma}(0) \left[ \left( \frac{\partial}{\partial z_i} U(\sqrt{\tilde{A}(0)} z) \right) \chi(\varepsilon z) + U(\sqrt{\tilde{A}(0)} z) \frac{\partial}{\partial z_i} \chi(\varepsilon z) \right] \\
&\quad \times \frac{\partial}{\partial z_i} \frac{\partial}{\partial y_1} \left[ \tilde{\gamma}(y) U_\varepsilon(\sqrt{\tilde{A}(y)} \tilde{\mathcal{E}}(y, \varepsilon z)) \chi(\tilde{\mathcal{E}}(y, \varepsilon z)) \right] \Big|_{y=0} d\eta + O(\varepsilon) \\
&= \int_{\mathbb{R}^n} \tilde{c}(\varepsilon z) \tilde{\gamma}(0) \frac{\partial}{\partial z_i} \left( U(\sqrt{\tilde{A}(0)} z) \right) \left\{ \frac{\partial}{\partial y_1} \tilde{\gamma}(y) \Big|_{y=0} \frac{\partial}{\partial z_i} U(\sqrt{\tilde{A}(0)} z) \right. \\
&\quad \left. + \tilde{\gamma}(0) \frac{\partial}{\partial z_i} U(\sqrt{\tilde{A}(0)} z) \frac{\partial}{\partial y_1} \chi(\tilde{\mathcal{E}}(y, \varepsilon z)) \Big|_{y=0} \right. \\
&\quad \left. + \tilde{\gamma}(0) \frac{\partial}{\partial y_1} \frac{\partial}{\partial z_i} U_\varepsilon(\sqrt{\tilde{A}(y)} \tilde{\mathcal{E}}(y, \varepsilon z)) \Big|_{y=0} \right\} dz + O(\varepsilon) =: D_1 + D_2 + D_3 + O(\varepsilon).
\end{aligned}$$

Expanding

$$\tilde{c}(\varepsilon z) = \tilde{c}(0) + \varepsilon \frac{\partial \tilde{c}}{\partial z_k}(0) z_k + O(\varepsilon^2 |z|)$$

we get

$$D_1 = \frac{1}{2} \frac{\partial}{\partial y_1} (\tilde{\gamma}(y))^2 \Big|_{y=0} \tilde{c}(0) \int_{\mathbb{R}^n} |\nabla_z \left( U(\sqrt{\tilde{A}(0)} z) \right)|^2 dz + O(\varepsilon)$$

and, by (2.7),

$$\begin{aligned}
D_2 &= \tilde{\gamma}^2(0) \int_{\mathbb{R}^n} \tilde{c}(\varepsilon z) |\nabla_z U(\sqrt{\tilde{A}(0)} z)|^2 \chi'(\varepsilon |z|) \frac{z_k}{|z|} \frac{\partial}{\partial y_1} \tilde{\mathcal{E}}_k(y, \varepsilon z) \Big|_{y=0} dz = \\
&= -\tilde{\gamma}^2(0) \tilde{c}(0) \int_{\mathbb{R}^n} |\nabla_z U(\sqrt{\tilde{A}(0)} z)|^2 \chi'(\varepsilon |z|) \frac{z_1}{|z|} dz + O(\varepsilon) = O(\varepsilon),
\end{aligned}$$

since the last integral is zero by symmetry reasons. At this point we observe that

$$\begin{aligned}
\frac{\partial U}{\partial z_1}(\sqrt{\tilde{A}(0)} z) &= U'(\sqrt{\tilde{A}(0)} z) \frac{z_1}{|z|}, \\
\frac{\partial}{\partial z_1} \left( \frac{U'(\sqrt{\tilde{A}(0)} z)}{|z|} \right) &= \left( \frac{\sqrt{\tilde{A}(0)} U''(z)}{|z|^2} - \frac{U'(z)}{|z|^3} \right) z_1,
\end{aligned}$$

where, abusing notation, we write

$$\frac{\partial U}{\partial z_1}(\sqrt{\tilde{A}(0)} z) = \frac{\partial}{\partial \eta_1} U(\eta) \Big|_{\eta = \sqrt{\tilde{A}(0)} z}.$$

In the same spirit, in the following we write

$$\frac{\partial^2 U}{\partial z_i \partial z_j}(\sqrt{\tilde{A}(0)} z) = \frac{\partial^2}{\partial \eta_i \partial \eta_j} U(\eta) \Big|_{\eta = \sqrt{\tilde{A}(0)} z}.$$

We have

$$\begin{aligned}
\frac{D_3}{\tilde{\gamma}^2(0)} &= \int_{\mathbb{R}^n} \tilde{c}(\varepsilon z) \frac{\partial}{\partial z_i} \left( U(\sqrt{\tilde{A}(0)}z) \right) \frac{\partial}{\partial z_i} \frac{\partial}{\partial y_1} U_\varepsilon(\sqrt{\tilde{A}(y)}\tilde{\mathcal{E}}(y, \varepsilon z)) \Big|_{y=0} dz \\
&= \int_{\mathbb{R}^n} \tilde{c}(\varepsilon z) \frac{\partial}{\partial z_i} \left( U(\sqrt{\tilde{A}(0)}z) \right) \frac{\partial}{\partial z_i} \left\{ \frac{\partial U}{\partial z_k}(\sqrt{\tilde{A}(0)}z) \left( \frac{\partial \sqrt{\tilde{A}(y)}}{\partial y_1} \Big|_{y=0} z_k + \frac{\sqrt{\tilde{A}(0)}}{\varepsilon} \frac{\partial}{\partial y_1} \tilde{\mathcal{E}}(y, \varepsilon z) \Big|_{y=0} \right) \right\} dz \\
&= \int_{\mathbb{R}^n} \tilde{c}(\varepsilon z) \frac{\partial \sqrt{\tilde{A}(y)}}{\partial y_1} \Big|_{y=0} \frac{\partial}{\partial z_i} \left( U(\sqrt{\tilde{A}(0)}z) \right) \frac{\partial}{\partial z_i} \left( \frac{\partial U}{\partial z_k}(\sqrt{\tilde{A}(0)}z) z_k \right) dz \\
&\quad - \frac{1}{\varepsilon} \int_{\mathbb{R}^n} \tilde{c}(\varepsilon z) \tilde{A}(0) \frac{\partial U}{\partial z_i}(\sqrt{\tilde{A}(0)}z) \frac{\partial}{\partial z_i} \left( \frac{\partial U}{\partial z_k}(\sqrt{\tilde{A}(0)}z) \delta_{1k} \right) dz + O(\varepsilon) \\
&= \int_{\mathbb{R}^n} \tilde{c}(\varepsilon z) \frac{\partial \sqrt{\tilde{A}(y)}}{\partial y_1} \Big|_{y=0} \frac{\partial}{\partial z_i} \left( U(\sqrt{\tilde{A}(0)}z) \right) \frac{\partial U}{\partial z_i}(\sqrt{\tilde{A}(0)}z) dz \\
&\quad + \int_{\mathbb{R}^n} \tilde{c}(\varepsilon z) \frac{\partial \sqrt{\tilde{A}(y)}}{\partial y_1} \Big|_{y=0} \sqrt{\tilde{A}(0)} \frac{\partial}{\partial z_i} \left( U(\sqrt{\tilde{A}(0)}z) \right) \frac{\partial^2 U}{\partial z_i \partial z_k}(\sqrt{\tilde{A}(0)}z) z_k dz \\
&\quad - \frac{1}{\varepsilon} \int_{\mathbb{R}^n} \tilde{c}(\varepsilon z) \tilde{A}(0) \frac{\partial U}{\partial z_i}(\sqrt{\tilde{A}(0)}z) \frac{\partial}{\partial z_i} \left( \frac{\partial U}{\partial z_1}(\sqrt{\tilde{A}(0)}z) \right) dz + O(\varepsilon) \\
&= \frac{1}{2} \int_{\mathbb{R}^n} \tilde{c}(0) \frac{\partial}{\partial y_1} \left| \nabla_z U(\sqrt{\tilde{A}(y)}z) \right|^2 \Big|_{y=0} dz \\
&\quad - \frac{1}{\varepsilon} \int_{\mathbb{R}^n} \tilde{c}(\varepsilon z) \tilde{A}(0) \frac{\partial U}{\partial z_i}(\sqrt{\tilde{A}(0)}z) \frac{\partial}{\partial z_i} \left( \frac{\partial U}{\partial z_1}(\sqrt{\tilde{A}(0)}z) \right) dz + O(\varepsilon).
\end{aligned}$$

Moreover, expanding  $\tilde{c}$  around  $z = 0$ ,

$$\begin{aligned}
& -\frac{1}{\varepsilon} \int_{\mathbb{R}^n} \tilde{c}(\varepsilon z) \tilde{A}(0) \frac{\partial U}{\partial z_i}(\sqrt{\tilde{A}(0)}z) \frac{\partial}{\partial z_i} \left( \frac{\partial U}{\partial z_1}(\sqrt{\tilde{A}(0)}z) \right) dz \\
& = -\frac{1}{\varepsilon} \int_{\mathbb{R}^n} \tilde{c}(0) \tilde{A}(0) \frac{\partial U}{\partial z_i}(\sqrt{\tilde{A}(0)}z) \frac{\partial}{\partial z_i} \left( \frac{\partial U}{\partial z_1}(\sqrt{\tilde{A}(0)}z) \right) dz \\
& \quad - \int_{\mathbb{R}^n} \frac{\partial \tilde{c}}{\partial z_k}(0) z_k \tilde{A}(0) \frac{\partial U}{\partial z_i}(\sqrt{\tilde{A}(0)}z) \frac{\partial}{\partial z_i} \left( \frac{\partial U}{\partial z_1}(\sqrt{\tilde{A}(0)}z) \right) dz + O(\varepsilon) \\
& = -\frac{1}{\varepsilon} \tilde{c}(0) \tilde{A}(0) \int_{\mathbb{R}^n} U'(\sqrt{\tilde{A}(0)}z) \frac{z_i}{|z|} \frac{\partial}{\partial z_i} \left( \frac{U'(\sqrt{\tilde{A}(0)}z)}{|z|} z_1 \right) \\
& \quad - \int_{\mathbb{R}^n} \frac{\partial \tilde{c}}{\partial z_k}(0) z_k \tilde{A}(0) U'(\sqrt{\tilde{A}(0)}z) \frac{z_i}{|z|} \frac{\partial}{\partial z_i} \left( \frac{U'(\sqrt{\tilde{A}(0)}z)}{|z|} z_1 \right) + O(\varepsilon) \\
& = - \int_{\mathbb{R}^n} \frac{\partial \tilde{c}}{\partial z_k}(0) \tilde{A}(0) \left( \frac{U'(\sqrt{\tilde{A}(0)}z)}{|z|} \right)^2 z_k z_1 dz \\
& \quad - \int_{\mathbb{R}^n} \frac{\partial \tilde{c}}{\partial z_k}(0) \tilde{A}(0) \frac{U'(\sqrt{\tilde{A}(0)}z)}{|z|} \left( \frac{\sqrt{\tilde{A}(0)} U''(z)}{|z|^2} - \frac{U'(z)}{|z|^3} \right) |z|^2 z_k z_1 dz + O(\varepsilon) \\
& = - \int_{\mathbb{R}^n} \frac{\partial \tilde{c}}{\partial z_1}(0) \tilde{A}(0) \left( \frac{U'(\sqrt{\tilde{A}(0)}z)}{|z|} \right)^2 z_1^2 dz \\
& \quad - \int_{\mathbb{R}^n} \frac{\partial \tilde{c}}{\partial z_1}(0) \tilde{A}(0) \frac{U'(\sqrt{\tilde{A}(0)}z)}{|z|} \left( \frac{\sqrt{\tilde{A}(0)} U''(z)}{|z|^2} - \frac{U'(z)}{|z|^3} \right) |z|^2 z_1^2 dz + O(\varepsilon) \\
& = -\frac{1}{2} \frac{\partial \tilde{c}}{\partial z_1}(0) \int_{\mathbb{R}^n} \frac{\partial}{\partial z_1} \left( \left| \nabla_z U(\sqrt{\tilde{A}(y)}z) \right|^2 \right) z_1 dz + O(\varepsilon) \\
& = \frac{1}{2} \int_{\mathbb{R}^n} \frac{\partial \tilde{c}}{\partial y_1}(y) \Big|_{y=0} \left| \nabla_z U(\sqrt{\tilde{A}(y)}z) \right|^2 dz + O(\varepsilon),
\end{aligned}$$

since the other integrals are zero by symmetry reasons. In conclusion, we have

$$I_1 = \frac{\partial}{\partial y_1} \left( \frac{1}{2} \int_{\mathbb{R}^n} c(\xi(y)) \left| \nabla_z V^{\xi(y)}(z) \right|^2 dz \right) \Big|_{y=0} + O(\varepsilon)$$



For the second term, by (2.8) and (2.9) we have, in an analogous way,

$$\begin{aligned}
(A.3) \quad I_2 &= \frac{1}{\varepsilon^n} \int_M a(x) W_{\varepsilon, \xi(y)} \frac{\partial}{\partial y_1} W_{\varepsilon, \xi(y)} d\mu_g \Big|_{y=0} \\
&= \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} |g_{\xi_0}(\eta)|^{\frac{1}{2}} a(\exp_{\xi_0} \eta) \tilde{\gamma}(0) U_\varepsilon(\sqrt{\tilde{A}(0)} \tilde{\mathcal{E}}(0, \eta)) \chi(\tilde{\mathcal{E}}(0, \eta)) \\
&\quad \times \frac{\partial}{\partial y_1} \left[ \tilde{\gamma}(y) U_\varepsilon(\sqrt{\tilde{A}(y)} \tilde{\mathcal{E}}(y, \eta)) \chi(\tilde{\mathcal{E}}(y, \eta)) \right] \Big|_{y=0} d\eta \\
&= \int_{\mathbb{R}^n} |g_{\xi_0}(\varepsilon z)|^{\frac{1}{2}} \tilde{a}(\varepsilon z) \tilde{\gamma}(0) U_\varepsilon(\sqrt{\tilde{A}(0)} \tilde{\mathcal{E}}(0, \varepsilon z)) \chi(\tilde{\mathcal{E}}(0, \varepsilon z)) \\
&\quad \times \frac{\partial}{\partial y_1} \left[ \tilde{\gamma}(y) U_\varepsilon(\sqrt{\tilde{A}(y)} \tilde{\mathcal{E}}(y, \varepsilon z)) \chi(\tilde{\mathcal{E}}(y, \varepsilon z)) \right] \Big|_{y=0} dz \\
&= \int_{\mathbb{R}^n} \tilde{a}(\varepsilon z) \tilde{\gamma}(0) U(\sqrt{\tilde{A}(0)} z) \\
&\quad \times \frac{\partial}{\partial y_1} \left[ \tilde{\gamma}(y) U_\varepsilon(\sqrt{\tilde{A}(y)} \tilde{\mathcal{E}}(y, \varepsilon z)) \chi(\tilde{\mathcal{E}}(y, \varepsilon z)) \right] \Big|_{y=0} dz + O(\varepsilon^2) = \\
&= \int_{\mathbb{R}^n} \tilde{a}(\varepsilon z) \tilde{\gamma}(0) U^2(\sqrt{\tilde{A}(0)} z) \frac{\partial}{\partial y_1} \tilde{\gamma}(y) \Big|_{y=0} dz \\
&\quad + \int_{\mathbb{R}^n} \tilde{a}(\varepsilon z) \tilde{\gamma}^2(0) U(\sqrt{\tilde{A}(0)} z) \frac{\partial}{\partial y_1} \left[ U_\varepsilon(\sqrt{\tilde{A}(y)} \tilde{\mathcal{E}}(y, \varepsilon z)) \right] \Big|_{y=0} dz + O(\varepsilon^2),
\end{aligned}$$

and for the last term we have

$$\begin{aligned}
(A.4) \quad &\int_{\mathbb{R}^n} \tilde{a}(\varepsilon z) \tilde{\gamma}^2(0) U(\sqrt{\tilde{A}(0)} z) \frac{\partial}{\partial y_1} \left[ U_\varepsilon(\sqrt{\tilde{A}(y)} \tilde{\mathcal{E}}(y, \varepsilon z)) \right] \Big|_{y=0} dz \\
&= \tilde{\gamma}^2(0) \frac{\partial}{\partial y_1} \sqrt{\tilde{A}(y)} \Big|_{y=0} \int_{\mathbb{R}^n} \tilde{a}(\varepsilon z) U(\sqrt{\tilde{A}(0)} z) \frac{\partial U}{\partial z_k}(\sqrt{\tilde{A}(0)} z) z_k dz \\
&\quad + \frac{\tilde{\gamma}^2(0) \sqrt{\tilde{A}(0)}}{\varepsilon} \int_{\mathbb{R}^n} \tilde{a}(\varepsilon z) U(\sqrt{\tilde{A}(0)}, z) \frac{\partial U}{\partial z_k}(\sqrt{\tilde{A}(0)} z) \frac{\partial}{\partial y_1} \tilde{\mathcal{E}}_k(y, \varepsilon z) \Big|_{y=0} dz \\
&= \tilde{\gamma}^2(0) \tilde{a}(0) \frac{\partial}{\partial y_1} \sqrt{\tilde{A}(y)} \Big|_{y=0} \int_{\mathbb{R}^n} U(\sqrt{\tilde{A}(0)} z) \frac{\partial U}{\partial z_k}(\sqrt{\tilde{A}(0)} z) z_k dz \\
&\quad - \frac{\tilde{\gamma}^2(0) \sqrt{\tilde{A}(0)}}{\varepsilon} \int_{\mathbb{R}^n} \tilde{a}(\varepsilon z) U(\sqrt{\tilde{A}(0)}, z) \frac{\partial U}{\partial z_1}(\sqrt{\tilde{A}(0)} z) dz + O(\varepsilon).
\end{aligned}$$

At this point we observe that  $\frac{\partial U}{\partial z_1}(\sqrt{\tilde{A}(0)} z) = U'(\sqrt{\tilde{A}(0)} z) \frac{z_1}{|z|}$  and, expanding

$$\tilde{a}(\varepsilon z) = \tilde{a}(0) + \varepsilon \frac{\partial \tilde{a}}{\partial z_k}(0) z_k + O(\varepsilon^2 |z|),$$

we obtain

$$\begin{aligned}
 (A.5) \quad & \frac{1}{\varepsilon} \int_{\mathbb{R}^n} \tilde{a}(\varepsilon z) U(\sqrt{\tilde{A}(0)}, z) U'(\sqrt{\tilde{A}(0)} z) \frac{z_1}{|z|} dz \\
 &= \frac{1}{\varepsilon} \int_{\mathbb{R}^n} \tilde{a}(0) U(\sqrt{\tilde{A}(0)}, z) U'(\sqrt{\tilde{A}(0)} z) \frac{z_1}{|z|} dz \\
 &+ \int_{\mathbb{R}^n} \frac{\partial \tilde{a}}{\partial z_k}(0) U(\sqrt{\tilde{A}(0)}, z) U'(\sqrt{\tilde{A}(0)} z) \frac{z_1 z_k}{|z|^2} dz + O(\varepsilon) \\
 &= \int_{\mathbb{R}^n} \frac{\partial \tilde{a}}{\partial z_1}(0) U(\sqrt{\tilde{A}(0)}, z) U'(\sqrt{\tilde{A}(0)} z) \frac{z_1^2}{|z|^2} dz + O(\varepsilon).
 \end{aligned}$$

In conclusion, from (A.3), (A.4) and (A.5) we get

$$\begin{aligned}
 I_2 &= \tilde{a}(0) \tilde{\gamma}(0) \left. \frac{\partial}{\partial y_1} \tilde{\gamma}(y) \right|_{y=0} \int_{\mathbb{R}^n} U^2(\sqrt{\tilde{A}(0)} z) dz \\
 &+ \tilde{\gamma}^2(0) \tilde{a}(0) \left. \frac{\partial}{\partial y_1} \sqrt{\tilde{A}(y)} \right|_{y=0} \sum_k \int_{\mathbb{R}^n} U(\sqrt{\tilde{A}(0)} z) U'(\sqrt{\tilde{A}(0)} z) \frac{z_k^2}{|z|^2} dz \\
 &- \tilde{\gamma}^2(0) \sqrt{\tilde{A}(0)} \frac{\partial \tilde{a}}{\partial z_1}(0) \int_{\mathbb{R}^n} U(\sqrt{\tilde{A}(0)}, z) U'(\sqrt{\tilde{A}(0)} z) \frac{z_1^2}{|z|^2} dz + O(\varepsilon) \\
 &= \frac{\partial}{\partial y_1} \left( \frac{1}{2} \int_{\mathbb{R}^n} a(\xi(y)) \left( V^{\xi(y)} \right)^2 dz \right) \Big|_{y=0} + O(\varepsilon)
 \end{aligned}$$

We argue in a similar way for  $I_3$  to complete the proof.  $\square$

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